

Bohl-Perron type stability theorems for linear difference equations with infinite delay

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Abstract

Relation between two properties of linear difference equations with infinite delay is investigated: (i) exponential stability, (ii) ℓ^p -input ℓ^q -state stability (sometimes is called Perron's property). The latter means that solutions of the non-homogeneous equation with zero initial data belong to ℓ^q when non-homogeneous terms are in ℓ^p . It is assumed that at each moment the prehistory (the sequence of preceding states) belongs to some weighted ℓ^r -space with an exponentially fading weight (the phase space).

Our main result states that (i) \Leftrightarrow (ii) whenever $(p, q) \neq (1, \infty)$ and a certain boundedness condition on coefficients is fulfilled. This condition is sharp and ensures that, to some extent, exponential and ℓ^p -input ℓ^q -state stabilities does not depend on the choice of a phase space and parameters p and q , respectively. ℓ^1 -input ℓ^∞ -state stability corresponds to uniform stability. We provide some evidence that similar criteria should not be expected for non-fading memory spaces.

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1 Introduction

We consider systems of linear difference equation with infinite delay

$$x(n+1) = L(n)x_n + f(n), \quad n \geq 0, \quad (1.1)$$

which in particular include *Volterra difference systems*

$$x(n+1) = \sum_{k=-\infty}^n L(n, n-k) x(k) + f(n), \quad n \geq 0. \quad (1.2)$$

It is assumed that $x(\cdot)$ is a discrete function from \mathbb{Z} to a (real or complex) Banach space \mathcal{X} , $f(\cdot)$ is a function from $\mathbb{Z}^+ (= \mathbb{N} \cup \{0\})$ to \mathcal{X} . The notation $|\cdot|$ stands for the norm in \mathcal{X} .

By x_n we denote the semi-infinite prehistory sequence $\{x(n), x(n-1), \dots, x(n+m), \dots\}$, $m \leq 0$. We suppose that the initial conditions, i.e., the sequence $x_0 = \{x(n+m)\}_{m=-\infty}^0$, belongs to an

exponentially weighted ℓ^∞ -space \mathcal{B}^γ , which is called the phase space. More precisely, it is assumed that for certain $\gamma \in \mathbb{R}$

$$|x_0|_{\mathcal{B}^\gamma} := \sup_{m \leq 0} |x(m)| e^{\gamma m} < \infty$$

and that $L(n)$, $n \geq 0$, are bounded linear mappings from \mathcal{B}^γ to \mathcal{X} .

The aim of the paper is to study relations between uniform exponential stability, uniform stability, and ℓ^p -input ℓ^q -state stability (or shorter (ℓ^p, ℓ^q) -stability) of (1.1). The precise definitions are given in Section 2.2.

For ordinary differential equations with coefficient $a(t)$ satisfying

$$\int_t^{t+1} |a(\tau)| d\tau \leq M_1 < \infty, \quad (1.3)$$

boundedness of a solution of the initial value problem

$$x'(t) + a(t)x(t) = f(t), \quad x(0) = 0, \quad t \geq 0, \quad (1.4)$$

for any bounded on $[0, \infty)$ right hand side f implies exponential stability of the corresponding homogeneous equation $x'(t) + a(t)x(t) = 0$. This result goes back to Bohl [6] and then was reinvented by Perron [25]; the above relation is sometimes called the Perron property. The Bohl-Perron result was extended to arbitrary Banach phase spaces by M. Krein, see notes to [12, Chapter III]. The result was later generalized in the following two directions. On the one hand, “for any $f \in \mathbf{L}^\infty$ the solution $x \in \mathbf{L}^\infty$ ” can be substituted by “for any $f \in \mathfrak{B}_1$ the solution $x \in \mathfrak{B}_2$ ”, where \mathfrak{B}_1 and \mathfrak{B}_2 are some Banach spaces of functions (see e.g. [12, Problems III.10-16] and references therein). In the terminology of our paper this property is called $(\mathfrak{B}_1, \mathfrak{B}_2)$ -stability. On the other hand, the Perron property was studied for various equations, such as delay differential equations, impulsive delay differential equations and difference equations. For first order system of difference equations $z(n+1) = A(n)z(n) + s(n)$, $n \geq 0$, the relation between (ℓ^p, ℓ^q) -stability and exponential stability was considered in [28, 29] (according to [28], this theory goes back to [8]) and later in [1, 2, 24, 27, 31] (see Theorem 2.6 below and subsequent remarks).

The Perron property for higher order difference equations was studied in [3, 4, 5, 10]. The paper [10] deals with *Volterra difference systems with unbounded delay*

$$x(n+1) = \sum_{k=0}^n L(n, n-k) x(k) + f(n), \quad n \geq 0; \quad (1.5)$$

exponential stability is understood in the sense of uniform in n estimates on the fundamental (resolvent) matrix, the role of the spaces $\mathfrak{B}_{1,2}$ is played by exponentially weighted ℓ^1 and ℓ^∞ spaces. In [3, 4, 5], (ℓ^p, ℓ^q) -stability for usual (nonweighted) ℓ^p -spaces is considered, estimates on the fundamental matrix are obtained and then applied to stability and exponential stability of equations with finite prehistory $\{x(n)\}_{n=-N}^0$, $N < \infty$. The case of bounded delay and $1 \leq p = q \leq \infty$ is considered in [3, 4] (as well as the case $\mathfrak{B}_1 = \mathfrak{B}_2 = c_0$), the unbounded delay and $p = q = \infty$ in [5].

The problem of finding Bohl-Perron type stability criteria for difference systems with infinite delay naturally requires the phase space settings of [21, 22, 23] (see Section 2.1). *In the present paper we solve this problem in exponentially fading phase spaces \mathcal{B}^γ , $\gamma > 0$.* Following the differential version of [17, Chapter 7], we use the notion of *uniform exponential stability in \mathcal{B}^γ* , which is different

from stability properties considered in [3, 4, 5, 10] (and is more appropriate for the infinite delay case). The method is based on the reduction of (1.1) to a first order system with states in the phase space. For systems with bounded delay this method has been announced in [5]. The main difficulty is the fact that the (ℓ^p, ℓ^q) -stability property of (1.1) is weaker than that of the reduced first order system.

The main points of the present paper are:

1. Uniform exponential stability and uniform stability are characterized in terms of (ℓ^p, ℓ^q) -stability (Theorems 3.1 and 4.5). For the particular case of Volterra difference systems these results can be written in the following way. *Let $\gamma > 0$. Assume that either $p \neq 1$ or $q \neq \infty$. Then the homogeneous system associated with (1.2) is uniformly exponentially stable in \mathcal{B}^γ if and only if system (1.2) is (ℓ^p, ℓ^q) -stable and*

$$\text{there exists a positive integer } l \text{ such that } \sup_{n \geq 0} \sum_{k \geq l} e^{k\gamma} \|L(n, k)\|_{\mathcal{X} \rightarrow \mathcal{X}} < \infty. \quad (1.6)$$

The homogeneous system associated with (1.2) is uniformly stable in \mathcal{B}^γ if and only if (1.2) is (ℓ^1, ℓ^∞) -stable and (1.6) holds.

2. *It is an immediate corollary that for systems with bounded delay condition (1.6) can be omitted.* This is an exact analogue of the case of a first order difference system for which Bohl-Perron type criteria do not require any boundedness restrictions on the coefficients $A(n)$ (see [28, 29, 31] and Section 2.2 for details). For differential equation (1.4) assumption (1.3) cannot be omitted: generally, the boundedness of a solution for any bounded right hand side does not imply exponential stability (see [12, Section III.5.3]). Similarly, in the case of systems with infinite delay, some assumptions involving uniform boundedness of coefficients are essential in the Bohl-Perron criteria. *Remark 3.3 and Example 6.2 shows that, in some sense, (1.6) is the weakest possible assumption of such type.*
3. There are two other interesting corollaries. Namely, *under the condition of (1.6) or its more general version (3.2), (i) (ℓ^p, ℓ^q) -stability does not depend on p and q (excluding the case $(p, q) = (1, \infty)$), (ii) exponential stability in \mathcal{B}^δ does not depend on the choice of $\delta \in (0, \gamma]$.* Examples of Section 6 show that for systems with unbounded delay these statements are not valid without condition (1.6).
4. It is essential that we consider exponentially fading phase spaces \mathcal{B}^γ , $\gamma > 0$. Example 6.3 shows that the main results (Theorems 3.1 and 4.5) are not valid in the non-fading phase space \mathcal{B}^0 . Nevertheless, for uniform stability in \mathcal{B}^0 we give two sufficient conditions of Bohl-Perron type (Corollary 4.9).
5. Main results can be easily extended to exponentially fading phase spaces of ℓ^p type (see discussion in Section 7).

The paper is organized as follows. After introducing in Section 2 some notations and presenting known results (which will be required in the sequel), we formulate the criterion of uniform exponential stability (Theorem 3.1) and some of its corollaries in Section 3.1. The proof of Theorem 3.1 is postponed to Section 3.3 and is preceded by auxiliary propositions of Section 3.2, which constitute the technical core of our method. In Sections 4.1-4.2 the above scheme is mimicked for uniform and

(ℓ^1, ℓ^∞) stabilities in \mathcal{B}^γ with $\gamma > 0$. Section 4.3 is devoted to the more difficult question of uniform stability in \mathcal{B}^0 . The independence of exponential stability of the choice of a phase space is discussed in Section 5 with the use of the notion of *subdiagonal systems* (subdiagonal systems are equivalent to shifted Volterra systems with unbounded delay, however the shift affects essentially the property of (ℓ^p, ℓ^q) -stability). Section 6 involves all relevant examples demonstrating sharpness of theorems' conditions. Section 7 contains discussion and open problems, as well as some additional applications of the presented method.

Finally note that other aspects of stability and boundedness of difference systems with unbounded delay were studied e.g. in [7, 10, 14, 15, 16, 18, 19, 20, 21, 23, 26]. An extensive list of applications can be found e.g. in [18]. Most of these papers are devoted to Volterra difference systems with unbounded delay.

2 Preliminaries and notation

2.1 The phase space and auxiliary spaces

As usual, we denote by \mathbb{Z} , \mathbb{Z}^+ , and \mathbb{Z}^- the set of all integers, the set of all nonnegative integers, and the set of all nonpositive integers, respectively. We shall sometimes write \mathbb{Z}_τ^+ to denote the infinite interval of integer numbers in $[\tau, +\infty)$, so $\mathbb{N} = \mathbb{Z}_1^+$. We use the convention that the sum equals zero if the lower index exceeds the upper index

$$\sum_k^j = 0 \quad \text{for } j < k. \quad (2.1)$$

For a seminormed space \mathcal{U} with a seminorm $|\cdot|_{\mathcal{U}}$, let $\mathcal{S}(\mathcal{U})$ ($\mathcal{S}_\pm(\mathcal{U})$) denote the vector space of all functions $v : \mathbb{Z} \rightarrow \mathcal{U}$ (resp., $v : \mathbb{Z}^\pm \rightarrow \mathcal{U}$). We will also use the following standard spaces:

$$\begin{aligned} \ell^p(\mathcal{U}) &:= \ell^p(\mathbb{Z}^+, \mathcal{U}) = \left\{ v : \mathbb{Z}^+ \rightarrow \mathcal{U} : \|v\|_p^p := \sum_{n=0}^{\infty} |v(n)|_{\mathcal{U}}^p < \infty \right\}, \quad 1 \leq p < \infty, \\ \ell^\infty(\mathcal{U}) &:= \ell^\infty(\mathbb{Z}^+, \mathcal{U}) = \left\{ v : \mathbb{Z}^+ \rightarrow \mathcal{U} : \|v\|_\infty := \sup_{n \in \mathbb{Z}^+} |v(n)|_{\mathcal{U}} < \infty \right\}. \end{aligned}$$

Recall that the ℓ^p -spaces are Banach spaces if \mathcal{U} is a Banach space, and that they are connected by the continuous embedding

$$\ell^p(\mathcal{U}) \subseteq \ell^q(\mathcal{U}), \quad \|v\|_q \leq \|v\|_p \quad \text{if } 1 \leq p \leq q \leq \infty. \quad (2.2)$$

Let a (real or complex) Banach space \mathcal{X} with a norm $|\cdot|$ be our basic space. For a definition of the concept of a phase space we use the vector space $\mathcal{X}^{\mathbb{Z}^-}$ of semi-infinite tuples with elements in \mathcal{X} and indices in \mathbb{Z}^- . It is convenient to understand vectors of $\mathcal{X}^{\mathbb{Z}^-}$ as vector-columns. That is, $\varphi \in \mathcal{X}^{\mathbb{Z}^-}$ has the form

$$\varphi = \text{col} \left(\varphi^{[m]} \right)_{m=-\infty}^0 = \begin{pmatrix} \varphi^{[0]} \\ \varphi^{[-1]} \\ \dots \\ \varphi^{[m]} \\ \dots \end{pmatrix}, \quad \text{where } \varphi^{[m]} \in \mathcal{X}, \quad m \in \mathbb{Z}^-.$$

We will say that $\varphi^{[m]}$ is the m -th coordinate of φ . (The notation of [23], where function space $\mathcal{S}_-(\mathcal{X})$ is used instead of $\mathcal{X}^{\mathbb{Z}^-}$ for prehistory vectors, can be considered as standard, but it is inconvenient for purposes of Section 3.2 of the present paper.)

Our main objects are the system (1.1) of nonhomogeneous linear functional difference equations and the associated homogeneous system

$$x(n+1) = L(n)x_n, \quad n \in \mathbb{Z}^+. \quad (2.3)$$

In formula (1.1), $x(\cdot)$, $L(\cdot)$, x_\bullet , and $f(\cdot)$ have the following meaning:

- ◇ $x = x(\cdot) : \mathbb{Z} \rightarrow \mathcal{X}$, $f = f(\cdot) : \mathbb{Z}^+ \rightarrow \mathcal{X}$, $x_\bullet : \mathbb{Z} \rightarrow \mathcal{X}^{\mathbb{Z}^-}$.
- ◇ The value $x_n \in \mathcal{X}^{\mathbb{Z}^-}$ of the function x_\bullet at n is the prehistory of $x(n)$, i.e.,

$$x_n^{[m]} := x(n+m), \quad m \in \mathbb{Z}^-,$$

$$\text{and } x_n = \text{col}(x_n^{[m]})_{m=-\infty}^0 = \begin{pmatrix} x_n^{[0]} \\ x_n^{[-1]} \\ \dots \\ x_n^{[m]} \\ \dots \end{pmatrix} := \begin{pmatrix} x(n) \\ x(n-1) \\ \dots \\ x(n+m) \\ \dots \end{pmatrix}. \quad (2.4)$$

- ◇ For each $n \in \mathbb{Z}^+$, $L(n) : \text{Dom } L(n) \rightarrow \mathcal{X}$ is a linear map defined on a certain linear subspace $\text{Dom } L(n)$ of $\mathcal{X}^{\mathbb{Z}^-}$. It is assumed also that the operator valued function $L = L(\cdot)$ defines system (1.1) on a certain *phase space* in the sense explained below.

Definition 2.1. A linear subspace $\mathcal{B} \subseteq \mathcal{X}^{\mathbb{Z}^-}$ is called a *phase space* if for any $x : \mathbb{Z} \rightarrow \mathcal{X}$, the inclusion $x_0 \in \mathcal{B}$ implies $x_n \in \mathcal{B}$ for all $n \in \mathbb{Z}^+$.

An example of a phase space is the vector space \mathcal{B}_{fin} of all *finite vectors* of $\mathcal{X}^{\mathbb{Z}^-}$, that is, of all $\varphi \in \mathcal{X}^{\mathbb{Z}^-}$ such that $\varphi^{[-j]}$ is zero for j large enough. It is easy to see that $\mathcal{B}_{\text{fin}} \subseteq \mathcal{B}$ for any phase space \mathcal{B} . A phase space \mathcal{B} is usually assumed to be equipped with a semi-norm or a norm $|\cdot|_{\mathcal{B}}$ that satisfies certain axioms (see e.g. [22, 23]).

Let \mathcal{U}_1 be a seminormed subspace of a certain vector space \mathcal{U} and let \mathcal{U}_2 be a normed space. Let J be a linear mapping from a linear manifold $\text{Dom } J \subseteq \mathcal{U}$ to \mathcal{U}_2 . We write $J \in \mathcal{L}(\mathcal{U}_1, \mathcal{U}_2)$ and say that J is a bounded linear operator from \mathcal{U}_1 to \mathcal{U}_2 if $\mathcal{U}_1 \subseteq \text{Dom } J$ and $\|J\|_{\mathcal{U}_1 \rightarrow \mathcal{U}_2} := \sup\{|Jv|_{\mathcal{U}_2} : |v|_{\mathcal{U}_1} \leq 1\}$ is finite. If $\mathcal{U}_1 = \mathcal{U}_2$ is a normed space, we write $\mathcal{L}(\mathcal{U}_2) := \mathcal{L}(\mathcal{U}_1, \mathcal{U}_2)$.

Definition 2.2. We shall say that the operator valued function $L(\cdot)$ defines system (1.1) on a *phase space* \mathcal{B} if $\mathcal{B} \subseteq \text{Dom } L(n)$ for all $n \in \mathbb{Z}^+$. If the phase space \mathcal{B} is a seminormed space, we assume additionally that $L(n) \in \mathcal{L}(\mathcal{B}, \mathcal{X})$ for all $n \in \mathbb{Z}^+$.

We will use normed and seminormed phase spaces of the following types: Banach spaces \mathcal{B}^γ defined for $\gamma \in \mathbb{R}$ by

$$\mathcal{B}^\gamma := \{\varphi = \text{col}(\varphi^{[m]})_{m=-\infty}^0 \in \mathcal{X}^{\mathbb{Z}^-} : |\varphi|_{\mathcal{B}^\gamma} := \sup_{m \in \mathbb{Z}^-} |\varphi^{[m]}|e^{\gamma m} < \infty\},$$

and the seminormed linear spaces $\mathcal{B}_{[j,0]}^\gamma$ defined for $j \in \mathbb{Z}^-$ and $\gamma \in \mathbb{R}$ by

$$\mathcal{B}_{[j,0]}^\gamma = \mathcal{X}^{\mathbb{Z}^-}, \quad |\varphi|_{\mathcal{B}_{[j,0]}^\gamma} := \sup_{j \leq m \leq 0} |\varphi^{[m]}|e^{\gamma m}.$$

It is clear that $\mathcal{B}_{[j,0]}^\gamma$ is complete, that is, the quotient $\mathcal{B}_{[j,0]}^\gamma / |\cdot|_{\mathcal{B}_{[j,0]}^\gamma}$ is a Banach space.

For phase spaces of ℓ^p -type see Section 7.

2.2 Solutions of initial value problems, stability and first order systems

Let \mathcal{W} be an auxiliary Banach space. The zero vectors of spaces \mathcal{X} and \mathcal{W} are denoted by $0_{\mathcal{X}}$ and $0_{\mathcal{W}}$, respectively. The zero vector of the vector space $\mathcal{X}^{\mathbb{Z}^-}$ is denoted by $0_{\mathcal{B}}$. The zero function of the function space $\mathcal{S}_+(\mathcal{U})$ (and its subspaces $\ell^p(\mathcal{U})$) will be denoted by $\mathbf{0}$ for any choice of \mathcal{U} .

From now on we assume that the function L defines system (1.1) on a phase space \mathcal{B} . For any $(\tau, \varphi) \in \mathbb{Z}^+ \times \mathcal{B}$, there exists unique $x : \mathbb{Z} \rightarrow \mathcal{X}$ such that $x_{\tau} = \varphi$ and the relation (1.1) holds for all $n \geq \tau$. The function x is called a solution of (1.1) through (τ, φ) , and is denoted by $x(\cdot, \tau, \varphi; f)$. For each $n \in \mathbb{Z}$, $x_n(\tau, \varphi; f)$ is the prehistory vector-column generated by $x(j, \tau, \varphi; f)$, $-\infty < j \leq n$ in the way shown by (2.4).

Definition 2.3. *The nonhomogeneous system (1.1) is called ℓ^p -input ℓ^q -state stable ((ℓ^p, ℓ^q) -stable, in short) if $x(\cdot, 0, 0_{\mathcal{B}}; f) \in \ell^q(\mathcal{X})$ for any $f \in \ell^p(\mathcal{X})$.*

The following definition is a modification of standard ones, see e.g. [17, Section 7.2] and [4, 5].

Definition 2.4. *Assume that the function L defines system (1.1) on a seminormed phase space \mathcal{B} .*

(1) The system (2.3) is called uniformly exponentially stable (UES) in (the sense of) \mathcal{X} with respect to the phase space \mathcal{B} if there exist $K \geq 1$ and $\nu > 0$ such that

$$|x(n, \tau, \varphi; \mathbf{0})| \leq K e^{-\nu(n-\tau)} |\varphi|_{\mathcal{B}} \quad \text{for all } n, \tau \text{ such that } n \geq \tau \geq 0. \quad (2.5)$$

(If the phase space \mathcal{B} is fixed we will say in brief that the system is UES in \mathcal{X} .)

(2) The system (2.3) is called UES in (the sense of) \mathcal{B} if the \mathcal{X} -norm $|x(n, \tau, \varphi; \mathbf{0})|$ in (2.5) is replaced by the \mathcal{B} -seminorm $|x_n(\tau, \varphi; \mathbf{0})|_{\mathcal{B}}$.

We will use some stability results for a *first order difference system*

$$z(n+1) = A(n)z(n), \quad n \in \mathbb{Z}^+, \quad (2.6)$$

where $z : \mathbb{Z}^+ \rightarrow \mathcal{W}$, $A : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{W})$, and \mathcal{W} is a certain Banach space. For $\psi \in \mathcal{W}$ and a function $s : \mathbb{Z}^+ \rightarrow \mathcal{W}$, we denote by $z(\cdot, \tau, \psi; s) : \mathbb{Z}_{\tau}^+ \rightarrow \mathcal{W}$ the solution of the associated nonhomogeneous initial value problem

$$z(n+1) = A(n)z(n) + s(n), \quad n \geq \tau \geq 0, \quad (2.7)$$

$$z(\tau) = \psi, \quad \psi \in \mathcal{W}. \quad (2.8)$$

Definition 2.5. *(1) The homogeneous system (2.6) is called UES if there exist $K_1 \geq 1$ and $\nu_1 > 0$ such that the solution of (2.6), (2.8) satisfies*

$$|z(n, \tau, \psi; \mathbf{0})|_{\mathcal{W}} \leq K_1 e^{-\nu_1(n-\tau)} |\psi|_{\mathcal{W}} \quad \text{for all } n \geq \tau \geq 0. \quad (2.9)$$

(2) The nonhomogeneous system (2.7) is called (ℓ^p, ℓ^q) -stable if $z(\cdot, 0, 0_{\mathcal{W}}; s) \in \ell^q(\mathcal{W})$ for any $s \in \ell^p(\mathcal{W})$.

For first order systems the following criterion is known.

Theorem 2.6 ([31], see also [28, 29, 1] for particular cases). *Let $A : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{W})$, let $1 \leq p \leq q \leq \infty$, and let the pair (p, q) be distinct from $(1, \infty)$. Then the homogeneous system (2.6) is UES if and only if the associated nonhomogeneous system (2.7) is (ℓ^p, ℓ^q) -stable.*

To the best of our knowledge, the general case of Theorem 2.6 was first proved in [31]. For $p = q = \infty$, the theorem was obtained earlier in [29, Section 4]. But it is easy to see that for the more general case $1 < p \leq q = \infty$ the essential part of the theorem (the implication 'if') follows from [28, Section 4] (one can check that (2.7) is uniformly equicontrollable in the terms of [28, Section 2]). Note also that the remark at the end of [28] states that the result of [28, Section 4] is contained implicitly in [8].

By a different method based on [2] the case $1 \leq p = q \leq \infty$ was proved in [1, Corollary 5], formally under the additional assumption $\sup_{n \in \mathbb{Z}^+} \|A(n)\|_{\mathcal{W} \rightarrow \mathcal{W}} < \infty$ (see also [24, 27]). This additional assumption can be easily removed (see Remark 2.8). Note that the papers [1, 2, 24] study stability along with with exponential dichotomy.

The following statement is standard, cf. Proof of Theorem 2.2 in [31] and [24, Lemma 2.3] for the case of first order systems.

Proposition 2.7. *Assume that $1 \leq p, q \leq \infty$, $\gamma \in \mathbb{R}$, and function $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$ defines system (1.1). If (1.1) is (ℓ^p, ℓ^q) -stable, then*

$$\|x(\cdot, 0, 0_{\mathcal{B}}; f)\|_q \leq K_{p,q,L} \|f\|_p \quad (2.10)$$

for a certain constant $K_{p,q,L} \geq 1$ depending on L .

Proof. The linear operator $\Gamma : f \rightarrow x(\cdot, 0, 0_{\mathcal{B}}; f)$ is correctly defined as an operator from $\ell^p(\mathcal{X})$ to $\ell^q(\mathcal{X})$. It follows easily from (1.1) that Γ is closed. By the closed graph principle, Γ is bounded. \square

We need also an analogue of the above proposition for the first order system (2.7). Namely, if $1 \leq p, q \leq \infty$ and $A : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{W})$, then

$$(\ell^p, \ell^q)\text{-stability of (2.7) implies } \|z(\cdot, 0, 0_{\mathcal{W}}; s)\|_q \leq K_{p,q,A} \|s\|_p \quad (2.11)$$

with a certain constant $K_{p,q,A} \geq 1$. The proof is the same.

Remark 2.8. (1) *UE stability of (2.6) immediately implies $\sup_{n \in \mathbb{Z}^+} \|A(n)\|_{\mathcal{W} \rightarrow \mathcal{W}} < \infty$. Indeed, from (2.9), we have*

$$|A(\tau)\psi|_{\mathcal{W}} = |z(\tau + 1, \tau, \psi; \mathbf{0})|_{\mathcal{W}} \leq K_1 e^{-\nu_1} \|\psi\|_{\mathcal{W}}, \quad \psi \in \mathcal{W}, \quad \tau \in \mathbb{Z}^+.$$

So $\|A(n)\|_{\mathcal{W} \rightarrow \mathcal{W}} \leq K_1 e^{-\nu_1}$ for all $n \in \mathbb{Z}^+$.

(2) *On the other hand, (ℓ^p, ℓ^q) -stability of (2.7) also yields $\sup_{n \in \mathbb{Z}^+} \|A(n)\|_{\mathcal{W} \rightarrow \mathcal{W}} < \infty$. In fact, for $\psi \in \mathcal{W}$, consider $s \in \ell^p(\mathcal{W})$ defined by $s(k) = \delta_{k,n} \psi$, where $\delta_{k,n}$ is Kronecker's delta. Then $z(n + 1, 0, 0_{\mathcal{W}}; s) = \psi$, $s(n + 1) = 0_{\mathcal{W}}$, and therefore $z(n + 2, 0, 0_{\mathcal{W}}; s) = A(n + 1)\psi$. By (2.11),*

$$|A(n + 1)\psi|_{\mathcal{W}} = |z(n + 2, 0, 0_{\mathcal{W}}; s)|_{\mathcal{W}} \leq \|z(\cdot, 0, 0_{\mathcal{W}}; s)\|_q \leq K_{p,q,A} \|s\|_p = K_{p,q,A} \|\psi\|_{\mathcal{W}}.$$

2.3 Auxiliary operators and related notation

Let \mathcal{B} be a phase space. Let I stand for the identity operator in $\mathcal{X}^{\mathbb{Z}^-}$ and so for the identity operator in \mathcal{B} .

For functions $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}, \mathcal{X})$ and $g : \mathbb{Z}^+ \rightarrow \mathcal{B}$, we define the function Lg by

$$Lg : \mathbb{Z}^+ \rightarrow \mathcal{X}, \quad (Lg)(n) := L(n)g(n), \quad n \in \mathbb{Z}^+.$$

If C is a map from \mathcal{B} to \mathcal{X} , then the function $Cg : \mathbb{Z}^+ \rightarrow \mathcal{X}$ has the natural meaning of

$$(Cg)(n) = Cg(n), \quad n \in \mathbb{Z}^+.$$

For $m \in \mathbb{Z}^-$ and $g : \mathbb{Z}^+ \rightarrow \mathcal{B}^\gamma$, we will use the shortening $g^{[m]}$ for the function that maps $n \in \mathbb{Z}^+$ to the m -th coordinate $(g(n))^{[m]}$ of $g(n)$, i.e.,

$$g^{[m]} : \mathbb{Z}^+ \rightarrow \mathcal{X}, \quad g^{[m]}(n) := (g(n))^{[m]}. \quad (2.12)$$

Let us define the 'backward shift' operator $S : \mathcal{X}^{\mathbb{Z}^-} \rightarrow \mathcal{X}^{\mathbb{Z}^-}$ by

$$(S\varphi)^{[m]} := \begin{cases} 0_{\mathcal{X}}, & m = 0 \\ \varphi^{[m+1]}, & m \leq -1 \end{cases}, \quad \varphi \in \mathcal{X}^{\mathbb{Z}^-}.$$

The operators S^j , $j \in \mathbb{N}$, shift coordinates of φ on j units downward and supplement coordinates with indices from $1 - j$ to 0 by the zero-element $0_{\mathcal{X}}$. As usual, $S^0 = I$.

For $m_1 \in \{-\infty\} \cup \mathbb{Z}^-$ and $m_2 \in \mathbb{Z}^-$ such that $m_1 \leq m_2$, we define the projection operator $P_{[m_1, m_2]} : \mathcal{X}^{\mathbb{Z}^-} \rightarrow \mathcal{X}^{\mathbb{Z}^-}$ by

$$(P_{[m_1, m_2]}\varphi)^{[m]} = \begin{cases} \varphi^{[m]}, & m_1 \leq m \leq m_2 \\ 0_{\mathcal{X}}, & \text{otherwise} \end{cases}, \quad m \in \mathbb{Z}^-, \quad \varphi \in \mathcal{X}^{\mathbb{Z}^-}. \quad (2.13)$$

The operator $P_{[m_1, m_2]}$ saves the coordinates from m_1 to m_2 and nulls all other coordinates. If $m_1 = m_2 = m \in \mathbb{Z}^-$, we write

$$P_{\{m\}} := P_{[m, m]}.$$

We will use extensively the operator $P_{\{0\}}$ that maps $\text{col}(\varphi^{[0]}, \varphi^{[-1]}, \varphi^{[-2]}, \dots)$ to $\text{col}(\varphi^{[0]}, 0_{\mathcal{X}}, 0_{\mathcal{X}}, \dots)$.

Note that for any $\varphi \in \mathcal{B}^\gamma$,

$$|S^j\varphi|_{\mathcal{B}^\gamma} = e^{-j\gamma}|\varphi|_{\mathcal{B}^\gamma}, \quad j \in \mathbb{Z}^+, \quad (2.14)$$

$$|\varphi|_{\mathcal{B}^\gamma} = \max\{ |P_{\{0\}}\varphi|_{\mathcal{B}^\gamma}, |(I - P_{\{0\}})\varphi|_{\mathcal{B}^\gamma} \}. \quad (2.15)$$

For $j \in \mathbb{Z}^-$ we consider also the operators

$$E_j : \mathcal{X} \rightarrow \mathcal{X}^{\mathbb{Z}^-}, \quad (E_j\psi)^{[m]} = \begin{cases} \psi, & m = j \\ 0_{\mathcal{X}}, & m \neq j \end{cases}, \quad \psi \in \mathcal{X}, \quad m \in \mathbb{Z}^-. \quad (2.16)$$

Assume that $\text{Dom } L(n) \supseteq \mathcal{B}_{\text{fin}}$ for all n . Let us define components of the operator $L(n)$ by

$$L(n, k) : \mathcal{X} \rightarrow \mathcal{X}, \quad L(n, k) := L(n)E_{-k}, \quad k \in \mathbb{Z}^+. \quad (2.17)$$

Remark 2.9. Note that generally for $L(n) \in \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$, operators $L(n, k)$, $k \in \mathbb{Z}^+$, do not determine the operator $L(n)$ on \mathcal{B}^γ . As an example, one can take $\mathcal{X} = \mathbb{R}$, $\gamma = 0$, and $L(n)$ equal to any of Banach limits (see e.g. [13, Sec. II.4.22] for the definition). Then $L(n, k) = 0$ for all k , but $L(n) \text{col}(1, 1, \dots) = 1$.

However, $L(n, k)$ determine $L(n)$ on finite vector-columns in the following way: for any $\varphi \in \mathcal{B}_{\text{fin}}$,

$$L(n)\varphi = \begin{pmatrix} L(n, 0) & L(n, 1) & \dots & L(n, -m) & \dots \end{pmatrix} \begin{pmatrix} \varphi^{[0]} \\ \varphi^{[-1]} \\ \dots \\ \varphi^{[m]} \\ \dots \end{pmatrix} = \sum_{m : \varphi^{[m]} \neq 0_{\mathcal{X}}} L(n, -m)\varphi^{[m]}.$$

3 Exponential stability and (ℓ^p, ℓ^q) -stability

The main result on UE stability, Theorem 3.1, and some of its corollaries are presented in Section 3.1. The proof of Theorem 3.1 given in Section 3.3 is based on the method described in Section 3.2.

3.1 Main results

Recall that the projection operators $P_{[m_1, m_2]}$ were defined in Section 2.3.

Theorem 3.1. *Let $\gamma > 0$ and let $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$ define system (1.1). Assume that the pair (p, q) is such that*

$$1 \leq p \leq q \leq \infty \quad \text{and} \quad (p, q) \neq (1, \infty). \quad (3.1)$$

Then the following statements are equivalent:

- (i) *System (2.3) is UES in \mathcal{X} with respect to (w.r.t.) \mathcal{B}^γ .*
- (ii) *System (2.3) is UES in \mathcal{B}^γ .*
- (iii) *System (1.1) is (ℓ^p, ℓ^q) -stable and*

$$\text{there exists } m \in \mathbb{Z}^- \text{ such that } \|L(\cdot)P_{[-\infty, m]}\|_\infty := \sup_{n \in \mathbb{Z}^+} \|L(n)P_{[-\infty, m]}\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} < \infty. \quad (3.2)$$

In the case $p = q = \infty$, Theorem 3.1 was obtained in [5] under certain additional conditions. The method of [5] differs from the method of the present paper.

Remark 3.2. *The proof of Theorem 3.1 shows that if any of statements (i)-(iii) of Theorem 3.1 is fulfilled, then $\sup_{n \in \mathbb{Z}^+} \|L(n)\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} < \infty$.*

Remark 3.3. (1) *Simple Example 6.1 demonstrates that condition (3.2) in Theorem 3.1 cannot be omitted. More subtle Example 6.2 shows that condition (3.2) cannot be replaced by the less restrictive condition*

$$\sup_{n \in \mathbb{Z}^+} \|L(n)P_{[-\infty, m_n]}\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} < \infty \quad (3.3)$$

with non-positive m_n such that $\lim_{n \rightarrow \infty} m_n = -\infty$.

(2) *Consider the case when $\gamma \leq 0$ and \mathcal{X} is nontrivial (i.e., $\mathcal{X} \neq \{0_{\mathcal{X}}\}$). Then UE stability in \mathcal{B}^γ does not hold for any system of the form (1.1). This follows immediately from the definitions of \mathcal{B}^γ and UE stability. Example 6.3 shows that, in general, the implication (iii) \Rightarrow (i) is also not valid.*

Since UE-stability does not depend on the choice of p and q in the (ℓ^p, ℓ^q) -stability property we get the following.

Corollary 3.4. *Let $\gamma > 0$ and let a function $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^\gamma, X)$ define system (1.1). Assume that (3.2) holds. Then (ℓ^p, ℓ^q) -stability of (1.1) for a certain pair (p, q) satisfying (3.1) implies (ℓ^p, ℓ^q) -stability of (1.1) for all (p, q) satisfying (3.1).*

Remark 3.5. Example 6.1 shows that assumption (3.2) in Corollary 3.4 cannot be dropped, however (3.2) can be relaxed to the condition that

$$\text{there exists } m \in \mathbb{Z}^- \text{ such that } \sup_{n \geq -m+1} \|L(n)P_{[-n+1, m]}\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} < \infty. \quad (3.4)$$

Indeed, (ℓ^p, ℓ^q) -stability of (1.1) does not depend on the parts $L(n)P_{[-\infty, -n]}$ of the operators $L(n)$ (see also Section 5).

Definition 3.6. Assume that $L(n)$ is defined on \mathcal{B}_{fin} for all $n \in \mathbb{Z}^+$ and that $L(n, k) \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ for all $n, k \in \mathbb{Z}^+$. Then (1.1) is called a system of difference equations with bounded delay if there exists $m \in \mathbb{Z}^-$ such that

$$L(n)P_{[-\infty, m]} = 0 \quad \text{for all } n \in \mathbb{Z}^+. \quad (3.5)$$

If m is the largest (nonpositive) number such that (3.5) holds, then $|m|$ is called the order of the system (1.1).

If (1.1) is a system with bounded delay of order d , then it can be written in the form

$$x(n+1) = \sum_{k=0}^{d-1} L(n, k)x(n-k) + f(n), \quad n \in \mathbb{Z}_+,$$

and it can be considered on the whole vector space $\mathcal{X}^{\mathbb{Z}^-}$. Any of the spaces \mathcal{B}^γ or $\mathcal{B}_{[j, 0]}^0$, with $j \leq -d+1$ and $\gamma \in \mathbb{R}$, can be chosen as a phase space. Note that UE stability in \mathcal{X} does not depend on that choice due to the obvious equality

$$x(\cdot, 0, \varphi; f) = x(\cdot, 0, P_{[-d+1, 0]}\varphi; f), \quad \varphi \in \mathcal{X}^{\mathbb{Z}^-}. \quad (3.6)$$

Thus, Theorem 3.1 implies the following result.

Corollary 3.7. Let $\gamma \in \mathbb{R}$, $1 \leq p \leq q \leq \infty$, and the pair (p, q) be distinct from $(1, \infty)$. If (1.1) is a system with bounded delay of order d , then the following statements are equivalent:

- (i) System (2.3) is UES in \mathcal{X} with respect to \mathcal{B}^γ (or, equivalently, w.r.t. $\mathcal{B}_{[-d+1, 0]}^\gamma$).
- (ii) System (1.1) is (ℓ^p, ℓ^q) -stable.

Proof. For \mathcal{B}^γ with $\gamma > 0$, the corollary follows from Theorem 3.1 and the fact that (3.2) is fulfilled for any system with bounded delay. Now note that the \mathcal{B}^γ -norms are equivalent for all $\gamma \in \mathbb{R}$ on the subspace of $\varphi \in \mathcal{X}^{\mathbb{Z}^-}$ such that $\varphi^m = 0_{\mathcal{X}}$ for $m \leq -d$ (that is, on the range of $P_{[-d+1, 0]}$). Combining this and (3.6) completes the proof. \square

The connection between UE stability and (ℓ^p, ℓ^q) -stability of systems with bounded delay was considered in [3, 4, 5]. Corollary 3.7 remove the assumption $\sup_{n \in \mathbb{Z}^+} \|L(n)\|_{\mathcal{B}^0 \rightarrow \mathcal{X}} < \infty$ imposed in [3, 4, 5] and extends the results of these papers to the case $1 \leq p < q < \infty$.

3.2 Reduction of order, representation theorem, and auxiliary results

Let $\gamma \in \mathbb{R}$ and let $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$ define system (1.1) on \mathcal{B}^γ . In this subsection, we show that system (1.1) can be written as a system of first order difference equations in the space \mathcal{B}^γ .

Recall that the operators S , E_0 , and $P_{\{0\}}$ are defined in Section 2.3.

Let us define operators $D(n) : \mathcal{B}^\gamma \rightarrow \mathcal{B}^\gamma$ by

$$D(n) := E_0 L(n) + S, \quad n \in \mathbb{Z}^+. \quad (3.7)$$

It follows from $L(n) \in \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$ and (2.14) that operators $D(n)$ are bounded in \mathcal{B}^γ and

$$\|D(n)\|_{\mathcal{B}^\gamma \rightarrow \mathcal{B}^\gamma} = \max\{ \|L(n)\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}}, e^{-\gamma} \}. \quad (3.8)$$

Let y and g be functions from \mathbb{Z}^+ to \mathcal{B}^γ . Let $\varphi \in \mathcal{B}^\gamma$ and $\tau \in \mathbb{Z}^+$. Consider the initial value problem

$$y(n+1) = D(n)y(n) + g(n), \quad n \in \mathbb{Z}_\tau^+, \quad (3.9)$$

$$y(\tau) = \varphi, \quad (3.10)$$

and denote its solution by $y(\cdot, \tau, \varphi; g)$. Then $y(n) = x_n(\tau, \varphi; f)$ is the solution of (3.9), (3.10) with $g = E_0 f$. In other words,

$$x_n(\tau, \varphi; f) = y(n, \tau, \varphi; E_0 f), \quad n \in \mathbb{Z}_\tau^+. \quad (3.11)$$

Let us show that it is possible to express $y(\cdot, 0, 0_{\mathcal{B}}; g)$ with general $g : \mathbb{Z}^+ \rightarrow \mathcal{B}^\gamma$ in terms of solutions of (1.1) and shift operators. Recall that we use the sum convention (2.1) and that for $g : \mathbb{Z}^+ \rightarrow \mathcal{B}^\gamma$, the function that maps $n \in \mathbb{Z}^+$ to the m -th coordinate $(g(n))^{[m]}$ of $g(n)$ is denoted by $g^{[m]}$.

Proposition 3.8. *For any $g : \mathbb{Z}^+ \rightarrow \mathcal{B}^\gamma$,*

$$y(n, 0, 0_{\mathcal{B}}; g) = x_n(0, 0_{\mathcal{B}}; g^{[0]} + Lh) + h(n), \quad \text{where} \quad (3.12)$$

$$h(n) = (Hg)(n) := \sum_{k=0}^{n-1} S^{n-k-1} (I - P_{\{0\}})g(k), \quad n \in \mathbb{Z}^+. \quad (3.13)$$

Proof. For $n = 0$, one can see that $h(0) = x_0(0, 0_{\mathcal{B}}; g^{[0]} + Lh) = 0_{\mathcal{B}}$, and therefore (3.12) is trivial. For $n = 1$, (1.1) implies

$$x(1, 0, 0_{\mathcal{B}}; g^{[0]} + Lh) = L(0)0_{\mathcal{B}} + g^{[0]}(0) + L(0)h(0) = g^{[0]}(0).$$

So the vector-column $x_1(0, 0_{\mathcal{B}}; g^{[0]} + Lh)$ has 0-th coordinate equal to $g^{[0]}(0)$ and all other coordinates equal to zero, i.e.,

$$x_1(0, 0_{\mathcal{B}}; g^{[0]} + Lh) = E_0 g^{[0]}(0).$$

Since $h(1) := (I - P_{\{0\}})g(0)$, we see that

$$x_1(0, 0_{\mathcal{B}}; g^{[0]} + Lh) + h(1) = E_0 g^{[0]}(0) + (I - P_{\{0\}})g(0) = g(0) = y(1, 0, 0_{\mathcal{B}}; g),$$

and so (3.12) holds true for $n = 1$.

Let us assume (3.12) for certain $n \in \mathbb{N}$ and prove it for $n + 1$. First, note that

$$h(n + 1) = Sh(n) + (I - P_{\{0\}})g(n), \quad n \in \mathbb{Z}^+. \quad (3.14)$$

Indeed,

$$\begin{aligned} Sh(n) + (I - P_{\{0\}})g(n) &= S \sum_{k=0}^{n-1} S^{n-k-1} (I - P_{\{0\}})g(k) + (I - P_{\{0\}})g(n) = \\ &= \sum_{k=0}^{n-1} S^{n-k} (I - P_{\{0\}})g(k) + (I - P_{\{0\}})g(n) = \sum_{k=0}^n S^{n-k} (I - P_{\{0\}})g(k) = h(n + 1). \end{aligned}$$

From (3.11) and (3.9), one can get

$$D(n)x_n(0, 0_{\mathcal{B}}; g^{[0]} + Lh) = x_{n+1}(0, 0_{\mathcal{B}}; g^{[0]} + Lh) - E_0 [g^{[0]}(n) + L(n)h(n)]. \quad (3.15)$$

Now we substitute (3.12) which is assumed to be valid for n into (3.9) and get

$$y(n + 1, 0, 0_{\mathcal{B}}; g) = D(n)x_n(0, 0_{\mathcal{B}}; g^{[0]} + Lh) + D(n)h(n) + g(n). \quad (3.16)$$

Modifying the last two terms with the use of (3.7), we get

$$\begin{aligned} D(n)h(n) + g(n) &= E_0 L(n)h(n) + Sh(n) + g(n) = \\ &= E_0 L(n)h(n) + Sh(n) + E_0 g^{[0]}(n) + (I - P_{\{0\}})g(n). \end{aligned}$$

Equality (3.14) implies

$$\begin{aligned} D(n)h(n) + g(n) &= [E_0 L(n)h(n) + E_0 g^{[0]}(n)] + [Sh(n) + (I - P_{\{0\}})g(n)] = \\ &= E_0 [L(n)h(n) + g^{[0]}(n)] + h(n + 1). \end{aligned}$$

Substituting the last equality and (3.15) into (3.16), we get

$$y(n + 1, 0, 0_{\mathcal{B}}; g) = x_{n+1}(0, 0_{\mathcal{B}}; g^{[0]} + Lh) + h(n + 1).$$

This is equality (3.12) for $n + 1$. Induction completes the proof. \square

Proposition 3.9. *Let $\gamma > 0$, $1 \leq p \leq \infty$, $g \in \ell^p(\mathcal{B}^\gamma)$, and let h be the function defined in (3.13). Then $\|h\|_p \leq (1 - e^{-\gamma})^{-1} \|g\|_p$.*

Proof. Recall that $h(0) = 0$ and consider $n \in \mathbb{N}$. Then for $m \in \mathbb{Z}^-$, m -th coordinate of $h(n)$ can be written in the following way

$$h^{[m]}(n) = \sum_{k=0}^{n-1} (S^{n-k-1} (I - P_{\{0\}})g(k))^{[m]} = \sum_{k=0}^{n-1} \begin{cases} 0, & m \geq -n + k + 1 \\ g^{[m+n-k-1]}(k), & m \leq -n + k \end{cases}$$

Since nonzero terms in the last sum correspond to $k \in \mathbb{Z}^+$ such that $k \geq m + n$, we have

$$h^{[m]}(n) = \sum_{k=\max\{0, m+n\}}^{n-1} g^{[m+n-k-1]}(k). \quad (3.17)$$

(Due to the sum convention (2.1) this formula is also valid for $n = 0$).

Using the last formula, we estimate $|h(n)|_{\mathcal{B}^\gamma}$ and then $\|h\|_p$:

$$\begin{aligned}
|h(n)|_{\mathcal{B}^\gamma} &= \sup_{m \in \mathbb{Z}^-} e^{m\gamma} \left| \sum_{k=\max\{0, m+n\}}^{n-1} g^{[m+n-k-1]}(k) \right| \leq \\
&\leq \sup_{m \in \mathbb{Z}^-} \sum_{k=\max\{0, m+n\}}^{n-1} e^{(-n+k+1)\gamma} e^{(m+n-k-1)\gamma} |g^{[m+n-k-1]}(k)| \leq \\
&\leq \sup_{m \in \mathbb{Z}^-} \sum_{k=\max\{0, m+n\}}^{n-1} e^{(-n+k+1)\gamma} |g(k)|_{\mathcal{B}^\gamma} = \sum_{k=0}^{n-1} e^{-((n-k)-1)\gamma} |g(k)|_{\mathcal{B}^\gamma} = (\mathfrak{e} * \mathfrak{g})(n), \quad (3.18)
\end{aligned}$$

where $(\mathfrak{e} * \mathfrak{g})(\bullet) = \sum_{k=0}^{\bullet} \mathfrak{e}(\bullet - k) \mathfrak{g}(k)$ is the discrete convolution of the functions

$$\mathfrak{g} \in \ell^p(\mathbb{R}), \quad \mathfrak{g}(n) := |g(n)|_{\mathcal{B}^\gamma}, \quad \text{and} \quad \mathfrak{e} \in \ell^1(\mathbb{R}), \quad \mathfrak{e}(n) = \begin{cases} 0, & n = 0 \\ e^{-(n-1)\gamma}, & n \geq 1 \end{cases}. \quad (3.19)$$

Using Young's inequality for convolutions (see e.g. [13, Problem VI.11.10]), we get

$$\|h\|_p \leq \|\mathfrak{e} * \mathfrak{g}\|_p \leq \|\mathfrak{e}\|_1 \|\mathfrak{g}\|_p = (1 - e^{-\gamma})^{-1} \|g\|_p.$$

□

Proposition 3.10. *Let $1 \leq p \leq \infty$, $\gamma > 0$, and let $x_0 = [x(m)]_{m=-\infty}^0$ belong to \mathcal{B}^γ . Then*

$$x_\bullet \in \ell^p(\mathbb{Z}^+, \mathcal{B}^\gamma) \quad \text{if and only if} \quad x(\cdot) \in \ell^p(\mathbb{Z}^+, \mathcal{X}).$$

More precisely,

$$\|x(\cdot)\|_\infty \leq \|x_\bullet\|_\infty = \max\{|x_0|_{\mathcal{B}^\gamma}, \|x(\cdot)\|_\infty\}, \quad (3.20)$$

$$\|x(\cdot)\|_p^p \leq \|x_\bullet\|_p^p \leq \frac{1}{1 - e^{-p\gamma}} (|x_0|_{\mathcal{B}^\gamma}^p + \|x(\cdot)\|_p^p), \quad 1 \leq p < \infty. \quad (3.21)$$

Proof. Formula (3.20) and the first inequality in (3.21) are obvious. Let us prove the second inequality in (3.21). Note that $|x_{-1}|_{\mathcal{B}^\gamma} \leq e^\gamma |x_0|_{\mathcal{B}^\gamma}$ and for $n \in \mathbb{Z}^+$,

$$|x_n|_{\mathcal{B}^\gamma}^p = \sup_{m \in \mathbb{Z}^-} |e^{\gamma m} x(n+m)|^p \leq e^{p\gamma(-n-1)} |x_{-1}|_{\mathcal{B}^\gamma}^p + \sum_{m=-n}^0 e^{p\gamma m} |x(n+m)|^p.$$

Therefore,

$$\begin{aligned}
\sum_{n=0}^{+\infty} |x_n|_{\mathcal{B}^\gamma}^p &\leq |x_{-1}|_{\mathcal{B}^\gamma}^p \sum_{n=0}^{+\infty} e^{-p\gamma(n+1)} + \sum_{m=-\infty}^0 e^{p\gamma m} \sum_{n=-m}^{+\infty} |x(n+m)|^p = \\
&= \frac{e^{-p\gamma}}{1 - e^{-p\gamma}} |x_{-1}|_{\mathcal{B}^\gamma}^p + \frac{1}{1 - e^{-p\gamma}} \|x(\cdot)\|_p^p \leq \frac{1}{1 - e^{-p\gamma}} (|x_0|_{\mathcal{B}^\gamma}^p + \|x(\cdot)\|_p^p).
\end{aligned}$$

□

Remark 3.11. Clearly, if $\gamma = 0$, the proposition is valid only for $p = \infty$. In this case (3.20) still holds.

Note that UE stability of (2.3) in \mathcal{B}^γ coincides with UE stability (in the sense of Definition 2.5) of the homogeneous system

$$y(n+1) = D(n)y(n), \quad y \in \mathcal{S}_+(\mathcal{B}^\gamma). \quad (3.22)$$

corresponding to (3.9). On the other hand system (2.3) is UES in \mathcal{X} if it is UES in \mathcal{B}^γ . If $\gamma > 0$ the converse is also true (cf. [17, Section 7.2] for the differential equation case).

Proposition 3.12. *Let $\gamma > 0$. Then the following statements are equivalent*

- (i) *System (2.3) is UES in \mathcal{X} w.r.t. \mathcal{B}^γ .*
- (ii) *System (2.3) is UES in \mathcal{B}^γ .*
- (iii) *System (3.22) is UES.*

Proof. We have only to prove that (2.5) implies (2.9) with $\mathcal{W} = \mathcal{B}^\gamma$ for the solution $y(n, \tau, \varphi; \mathbf{0})$ of (3.22).

Indeed,

$$|x(n, \tau, \varphi; \mathbf{0})| = |P_{\{0\}}y(n, \tau, \varphi; \mathbf{0})|_{\mathcal{B}^\gamma},$$

so (2.5) can be rewritten as

$$|P_{\{0\}}y(n, \tau, \varphi; \mathbf{0})|_{\mathcal{B}^\gamma} \leq K e^{-\nu(n-\tau)} |\varphi|_{\mathcal{B}^\gamma}, \quad n \geq \tau \geq 0,$$

(recall that $K \geq 1$ and $\nu > 0$). Note that

$$|(I - P_{\{0\}})y(n, \tau, \varphi; \mathbf{0})|_{\mathcal{B}^\gamma} = e^{-\gamma} |y(n-1, \tau, \varphi; \mathbf{0})|_{\mathcal{B}^\gamma}.$$

Using (2.15), we see that the assumption

$$|y(n-1, \tau, \varphi; \mathbf{0})|_{\mathcal{B}^\gamma} \leq K e^{-\nu_1(n-1-\tau)} |\varphi|_{\mathcal{B}^\gamma}, \quad \text{where } \nu_1 := \min\{\nu, \gamma\},$$

implies

$$|y(n, \tau, \varphi; \mathbf{0})|_{\mathcal{B}^\gamma} \leq \max\{ K e^{-\nu(n-\tau)} |\varphi|_{\mathcal{B}^\gamma}, K e^{-\gamma} e^{-\nu_1(n-1-\tau)} |\varphi|_{\mathcal{B}^\gamma} \} \leq K e^{-\nu_1(n-\tau)} |\varphi|_{\mathcal{B}^\gamma}.$$

Induction completes the proof. □

Recall that the operators $L(n, k)$ are defined in Section 2.3.

Proposition 3.13. *Let $\gamma \in \mathbb{R}$, $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$, and $1 \leq p, q \leq \infty$. Assume that (1.1) is (ℓ^p, ℓ^q) -stable. Then:*

- (i) *For any $k \geq 0$ and $n \geq k+1$,*

$$\|L(n, k)\|_{\mathcal{X} \rightarrow \mathcal{X}} \leq 2^k K_{p,q,L}^{k+1},$$

where $K_{p,q,L}$ is the constant introduced in Proposition 2.7.

(ii) For any $j \in \mathbb{Z}^-$,

$$\sup_{n \in \mathbb{Z}^+} \|L(n)P_{[j,0]}\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} < \infty. \quad (3.23)$$

Proof. (i) We use induction in k to prove that

$$\|L(n_0 + k, k)\|_{\mathcal{X} \rightarrow \mathcal{X}} \leq 2^k K_{p,q,L}^{k+1}. \quad (3.24)$$

for any $n_0 \in \mathbb{N}$ and $k \in \mathbb{Z}^+$.

Let $k = 0$. Then (3.24) can be obtained in the way shown in Remark 2.8 (2). Assume now that $k_1 \in \mathbb{N}$ and (3.24) holds for all $0 \leq k \leq k_1 - 1$.

For any $\psi \in \mathcal{X}$ and $n_0 \in \mathbb{N}$ we can choose $f \in \ell^p(\mathcal{X})$ such that

$$x_{n_0-1}(0, 0_{\mathcal{B}}; f) = 0_{\mathcal{B}}, \quad x_{n_0+k}(0, 0_{\mathcal{B}}; f) = E_{-k}\psi \quad \text{for } 0 \leq k \leq k_1. \quad (3.25)$$

Indeed, consider $f(\cdot)$ defined by

$$f(j) = 0_{\mathcal{X}} \quad \text{for } 0 \leq j \leq n_0 - 2, \quad (3.26)$$

$$f(n_0 - 1) = \psi, \quad (3.27)$$

$$f(n_0 + k) = -L(n_0 + k, k)\psi \quad \text{for } 0 \leq k \leq k_1 - 1, \quad (3.28)$$

$$f(n_0 + k) = 0_{\mathcal{X}} \quad \text{for } k \geq k_1. \quad (3.29)$$

Then it is easy to see that (3.25) holds. Note that

$$x(n_0 + k_1 + 1) = L(n_0 + k_1, k_1)\psi \quad \text{and} \quad |x(n_0 + k_1 + 1)| \leq \|x(\cdot, 0, 0_{\mathcal{B}}; f)\|_q.$$

Applying Proposition 2.7 to f defined by (3.26)–(3.29), we get

$$|L(n_0 + k_1, k_1)\psi| \leq \|x(\cdot, 0, 0_{\mathcal{B}}; f)\|_q \leq K_{p,q,L} \|f\|_p.$$

It follows from (2.2) that

$$\|f\|_p \leq \|f\|_1 = |\psi| + \sum_{k=0}^{k_1-1} |L(n_0 + k, k)\psi| \leq |\psi| \left(1 + \sum_{k=0}^{k_1-1} |L(n_0 + k, k)|_{\mathcal{X} \rightarrow \mathcal{X}} \right).$$

Combining the last two inequalities with (3.24) for $0 \leq k \leq k_1 - 1$ and the fact that $K_{p,q,L} \geq 1$, we get

$$\|L(n_0 + k_1, k_1)\|_{\mathcal{X} \rightarrow \mathcal{X}} \leq K_{p,q,L} \left(1 + \sum_{k=0}^{k_1-1} 2^k K_{p,q,L}^{k+1} \right) \leq K_{p,q,L}^{k_1+1} \left(1 + \sum_{k=0}^{k_1-1} 2^k \right) = 2^{k_1} K_{p,q,L}^{k_1+1}.$$

This completes the proof of (i).

(ii) The assertion $\sup_{n \in \mathbb{Z}^+} \|L(n)P_{[j,0]}\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} < \infty$ follows from the estimate

$$\sup_{n \geq -j+1} \|L(n)P_{[j,0]}\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} \leq K_{p,q,L} \frac{(2e^\gamma K_{p,q,L})^{-j+1} - 1}{2e^\gamma K_{p,q,L} - 1}. \quad (3.30)$$

Let us show that statement (i) implies (3.30). Note that for $n \geq -j + 1$ and $j \leq m \leq 0$,

$$|L(n)P_{\{m\}}\varphi| = |L(n, -m)\varphi^{[m]}| \leq 2^{-m}K_{p,q,L}^{-m+1}|\varphi^{[m]}| \leq 2^{-m}K_{p,q,L}^{-m+1}e^{-m\gamma}|\varphi|_{\mathcal{B}^\gamma}.$$

Hence,

$$|L(n)P_{[j,0]}\varphi| \leq \sum_{m=j}^0 |L(n)P_{\{m\}}\varphi| \leq |\varphi|_{\mathcal{B}^\gamma} \sum_{m=j}^0 2^{-m}e^{-m\gamma}K_{p,q,L}^{-m+1} \leq |\varphi|_{\mathcal{B}^\gamma} K_{p,q,L} \frac{(2e^\gamma K_{p,q,L})^{-j+1} - 1}{2e^\gamma K_{p,q,L} - 1}.$$

□

3.3 Proof of Theorem 3.1

The proof is based on Theorem 2.6 and the reduction of (1.1) to (3.9). The facts that **(i)** \Leftrightarrow **(ii)** and that **(i)** and **(ii)** are equivalent to UE stability of (3.22) are established in Proposition 3.12.

Let us prove that **(iii) implies UE stability of (3.22)**. Taking into account Theorem 2.6, it is enough to prove that **(iii)** implies

$$\|y(\cdot, 0, 0_{\mathcal{B}}; g)\|_q \leq C_1 \|g\|_p, \quad g \in \ell^p(\mathcal{B}^\gamma), \quad (3.31)$$

with a certain constant $C_1 = C_1(L) > 0$.

By Proposition 3.8,

$$\|y(\cdot, 0, 0_{\mathcal{B}}; g)\|_q \leq \|x_\bullet(0, 0_{\mathcal{B}}; g^0 + Lh)\|_q + \|h\|_q.$$

Applying Proposition 3.10 to the first term (note that $x_0 = 0_{\mathcal{B}}$) and inequality (2.2) to the second, we get

$$\|y(\cdot, 0, 0_{\mathcal{B}}; g)\|_q \leq C_2(q) \|x(\cdot, 0, 0_{\mathcal{B}}; g^0 + Lh)\|_q + \|h\|_p,$$

where $C_2(q) := (1 - e^{-q\gamma})^{-1/q}$ for $q < \infty$ and $C_2(\infty) := 1$. From (ℓ^p, ℓ^q) -stability and Proposition 2.7 we obtain

$$\|y(\cdot, 0, 0_{\mathcal{B}}; g)\|_q \leq C_2(q) K_{p,q,L} \|g^0\|_p + C_2(q) K_{p,q,L} \|Lh\|_p + \|h\|_p.$$

Note that (3.2) and Proposition 3.13 (ii) imply $\|L(\cdot)\|_\infty < \infty$. This and Proposition 3.9 yield

$$\begin{aligned} \|y(\cdot, 0, 0_{\mathcal{B}}; g)\|_q &\leq C_2(q) K_{p,q,L} \|g^0\|_p + C_2(q) K_{p,q,L} \|L\|_\infty \|h\|_p + \|h\|_p \leq \\ &\leq C_2(q) K_{p,q,L} \|g\|_p + [C_2(q) K_{p,q,L} \|L\|_\infty + 1] (1 - e^{-\gamma})^{-1} \|g\|_p. \end{aligned}$$

This completes the proof of (3.31).

Let us show that **UE stability of (3.22) implies (iii)**. It follows from Theorem 2.6 and formula (3.11) that UE stability of (3.22) implies $x_\bullet(0, 0_{\mathcal{B}}; f) \in \ell^q(\mathcal{X})$ for any $f \in \ell^p(\mathcal{X})$. So system (1.1) is (ℓ^p, ℓ^q) -stable.

Finally, note that UE stability of (3.22) implies $\|L(\cdot)\|_\infty < \infty$ and so implies (3.2) for every $m \in \mathbb{Z}^-$. Indeed, we see from Remark 2.8 (1) that **(ii)** yields $\|D(\cdot)\|_\infty < \infty$ (the operators $D(n)$ are defined by (3.7)). Now (3.8) implies $\|L(\cdot)\|_\infty < \infty$. This completes the proof.

4 Uniform stability and (ℓ^1, ℓ^∞) -stability

In this section we prove that in the case when $p = 1$ and $q = \infty$, the theorem analogous to Theorem 3.1 is valid with uniform stability instead of UE-stability. We also consider the more difficult case when $\gamma = 0$. As before, our method is based on the reduction of system (1.1) to the first order system (2.7).

4.1 Preliminaries

Let a function $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}, \mathcal{X})$ define system (1.1) on a seminormed phase space \mathcal{B} . Then homogeneous system (2.3) is called *uniformly stable (US, in short) in \mathcal{X} w.r.t. the phase space \mathcal{B}* if (2.5) holds with $\nu = 0$. (If the phase space \mathcal{B} is fixed we will say in brief that a system is US in \mathcal{X} .)

Uniform stability in \mathcal{B} of (2.3) and *uniform stability* of the first order system (2.6) are defined in the similar way placing $\nu = 0$ in Definitions 2.4 (2) and 2.5 (1), respectively.

We will use the following result concerning first order systems.

Theorem 4.1 (cf. [1]). *Let a function $A : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{W})$ define the first order system (2.7). Then the corresponding homogeneous system (2.6) is US if and only if system (2.7) is (ℓ^1, ℓ^∞) -stable.*

Remark 4.2. *This result was obtained in [1, Theorem 6], formally under the additional assumption $\sup_{n \in \mathbb{Z}^+} \|A(n)\|_{\mathcal{W} \rightarrow \mathcal{W}} < \infty$. This additional assumption can be easily dropped in the way shown in Remark 2.8. In fact, uniform stability of (2.6), as well as (ℓ^1, ℓ^∞) -stability of (2.7), implies $\sup_{n \in \mathbb{Z}^+} \|A(n)\|_{\mathcal{W} \rightarrow \mathcal{W}} < \infty$.*

Proposition 4.3. *Let $\gamma \geq 0$ and let $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$. Then the following statements are equivalent:*

- (i) *System (2.3) is US in \mathcal{X} w.r.t. \mathcal{B}^γ .*
- (ii) *System (2.3) is US in \mathcal{B}^γ .*
- (iii) *System (3.22) is US.*

The proof is the same as that of Proposition 3.12 if we set $\nu = \nu_1 = 0$.

Proposition 4.4. *Let $\gamma \geq 0$ and let $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$. Then any of assertions (i)-(iii) of Proposition 4.3 implies $\sup_{n \in \mathbb{Z}^+} \|L(n)\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} < \infty$.*

The proposition follows from assertion (iii) of Proposition 4.3, Remark 4.2, and formula (3.8).

4.2 Uniform stability in the phase space \mathcal{B}^γ with $\gamma > 0$.

Theorem 4.5. *Let $\gamma > 0$ and let a function $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$ define system (1.1). Then the following statements are equivalent*

- (i) *System (2.3) is US in \mathcal{B}^γ (or, equivalently, in \mathcal{X} w.r.t. \mathcal{B}^γ).*
- (ii) *System (1.1) is (ℓ^1, ℓ^∞) -stable and condition (3.2) is fulfilled.*

The proof of Theorem 4.5 is similar to the proof of Theorem 3.1 with the use of Proposition 4.3 instead of Proposition 3.12, Remark 3.11 instead of Proposition 3.10, Theorem 4.1 instead of Theorem 2.6, and Remark 4.2 instead of Remark 2.8 (1).

Remark 4.6. *Example 6.2 and Proposition 4.4 show that condition (3.2) in Theorem 4.5 cannot be replaced by (3.3) with $\{m_n\}_1^\infty$ such that $\lim m_n = -\infty$.*

Recall that systems with bounded delay are defined in Section 3.1.

Corollary 4.7. *Let $\gamma \in \mathbb{R}$. If (1.1) is a system with bounded delay of order d , then the following statements are equivalent:*

- (i) *System (2.3) is US in \mathcal{X} with respect to \mathcal{B}^γ (or, equivalently, w.r.t. $\mathcal{B}_{[-d+1,0]}^\gamma$).*
- (ii) *System (1.1) is (ℓ^1, ℓ^∞) -stable.*

The proof is similar to that of Corollary 3.7.

4.3 The case $\gamma = 0$

Example 6.4 shows that in general Theorem 4.5 is not valid for $\gamma = 0$. In this subsection we give several results on uniform stability of (2.3) in \mathcal{B}^0 .

Let $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^0, \mathcal{X})$. Recall that the operator $H : \mathcal{S}_+(\mathcal{B}^0) \rightarrow \mathcal{S}_+(\mathcal{B}^0)$ is defined in (3.13). Let us define the operator $M_L : \mathcal{S}_+(\mathcal{B}^0) \rightarrow \mathcal{S}_+(\mathcal{X})$ by

$$(M_L g)(n) := L(n)g(n), \quad n \in \mathbb{Z}^+, \quad g \in \mathcal{S}_+(\mathcal{B}^0).$$

As in the proof of Proposition 2.7, Γ is the linear operator from $\mathcal{S}_+(\mathcal{X})$ to $\mathcal{S}_+(\mathcal{X})$ defined by

$$\Gamma : f \mapsto x(\cdot, 0, 0_{\mathcal{B}}; f).$$

In the following theorem we use the product $\Gamma M_L H$ of the three above defined operators and the image $(\Gamma M_L H) \ell^1(\mathcal{B}^0)$ of the space $\ell^1(\mathbb{Z}^+, \mathcal{B}^0)$ under the operator $\Gamma M_L H$. The image is understood in the usual sense

$$(\Gamma M_L H) \ell^1(\mathcal{B}^0) := \{ x(\cdot) \in \mathcal{S}_+(\mathcal{X}) : x = \Gamma M_L H g \text{ for some } g \in \ell^1(\mathcal{B}^0) \}.$$

Theorem 4.8. *Let a function $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^0, \mathcal{X})$ define system (1.1). Then the following statements are equivalent*

- (i) *System (2.3) is US in \mathcal{B}^0 (or, equivalently, in \mathcal{X} w.r.t. \mathcal{B}^0).*
- (ii) *System (1.1) is (ℓ^1, ℓ^∞) -stable and $(\Gamma M_L H) \ell^1(\mathcal{B}^0) \subseteq \ell^\infty(\mathcal{X})$.*

Proof. (i) \Rightarrow (ii). By Proposition 4.3 and Theorem 4.1, assertion (i) implies that the first order system (3.9) considered in the space \mathcal{B}^0 is (ℓ^1, ℓ^∞) -stable. This and (3.11) yield (ℓ^1, ℓ^∞) -stability of (1.1). In particular,

$$\Gamma g^{[0]} \in \ell^\infty(\mathcal{X}) \quad \text{for any } g \in \ell^1(\mathcal{B}^0). \quad (4.1)$$

For $g \in \ell^1(\mathcal{B}^0)$, equality (3.12), (ℓ^1, ℓ^∞) -stability of (3.9), and $|x(n, 0, 0_{\mathcal{B}}; f)| \leq |x_n(0, 0_{\mathcal{B}}; f)|_{\mathcal{B}^0}$ imply

$$\Gamma(g^{[0]} + M_L H g) = x(\cdot, 0, 0_{\mathcal{B}}; g^{[0]} + M_L H g) \in \ell^\infty(\mathcal{X}).$$

Taking into account (4.1), we get $\Gamma M_L H g \in \ell^\infty(\mathcal{X})$.

Inverting the above arguments with the use of Remark 3.11 one can get (ii) \Rightarrow (i). □

The condition $(\Gamma M_L H) \ell^1(\mathcal{B}^0) \subseteq \ell^\infty(\mathcal{X})$ has a complicated nature. The following sufficient conditions are simpler for understanding.

Corollary 4.9. *Let a function $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^0, \mathcal{X})$ define system (1.1). Then each of the following two conditions is sufficient for (2.3) to be US in \mathcal{B}^0 :*

(i) *System (1.1) is $(\ell^\infty, \ell^\infty)$ -stable and*

$$\text{there exists } m \in \mathbb{Z}^- \text{ such that } \|L(\cdot)P_{[-\infty, m]}\|_\infty := \sup_{n \in \mathbb{Z}^+} \|L(n)P_{[-\infty, m]}\|_{\mathcal{B}^0 \rightarrow \mathcal{X}} < \infty. \quad (4.2)$$

(ii) *System (1.1) is (ℓ^1, ℓ^∞) -stable and*

$$\text{there exists } m \in \mathbb{Z}^- \text{ such that } \|L(\cdot)P_{[-\infty, m]}\|_1 := \sum_{n=0}^{\infty} \|L(n)P_{[-\infty, m]}\|_{\mathcal{B}^0 \rightarrow \mathcal{X}} < \infty. \quad (4.3)$$

Proof. In the both cases we check assertion (ii) of Theorem 4.8 using the following estimates

$$\|Hg\|_\infty := \sup_{n \in \mathbb{Z}^+} |(Hg)(n)|_{\mathcal{B}^0} \leq \|g\|_1, \quad (4.4)$$

$$\|P_{[j, 0]}(Hg)(\cdot)\|_1 := \sum_{n=0}^{\infty} |P_{[j, 0]}(Hg)(n)|_{\mathcal{B}^0} \leq (-j) \|g\|_1, \quad j \in \mathbb{Z}^-. \quad (4.5)$$

The first one follows immediately from (3.17). For the proof of the second, let us note that (3.17) implies

$$|P_{[j, 0]}(Hg)(n)|_{\mathcal{B}^0} = \max_{j \leq m \leq 0} |(Hg)^{[m]}(n)| \leq \sum_{k=\max\{0, n+j\}}^{n-1} |g(k)|_{\mathcal{B}^0}.$$

Hence (recall the sum convention (2.1))

$$\sum_{n=0}^{\infty} |P_{[j, 0]}(Hg)(n)|_{\mathcal{B}^0} \leq \sum_{n=1}^{\infty} \sum_{k=\max\{0, n+j\}}^{n-1} |g(k)|_{\mathcal{B}^0}.$$

Since each $g(n)$ participates in the last sum at most $(-j)$ times, we get (4.5).

(i) implies uniform stability of (2.3). (ℓ^1, ℓ^∞) -stability follows from $(\ell^\infty, \ell^\infty)$ -stability.

Formulae (3.23) and (4.2) imply that $\|L\|_\infty := \sup_{n \in \mathbb{Z}^+} \|L(n)\|_{\mathcal{B}^0 \rightarrow \mathcal{X}} < \infty$. From (4.4) we get

$$\|M_L Hg\|_\infty \leq \|L\|_\infty \|Hg\|_\infty \leq \|L\|_\infty \|g\|_1.$$

So $M_L Hg \in \ell^\infty(\mathcal{X})$ for $g \in \ell^1(\mathcal{B}^0)$ and therefore $(\ell^\infty, \ell^\infty)$ -stability implies $\Gamma M_L Hg \in \ell^\infty(\mathcal{X})$.

(ii) implies uniform stability of (2.3). Let $g \in \ell^1(\mathcal{B}^0)$. Without loss of generality, we can assume $m \leq -1$ in (4.3). Taking $j = m + 1$, one can get from (4.5) and (3.23) that

$$\begin{aligned} \sum_{n=0}^{\infty} |L(n)P_{[m+1, 0]}(Hg)(n)| &\leq \|L(\cdot)P_{[m+1, 0]}\|_\infty \|P_{[m+1, 0]}(Hg)(\cdot)\|_1 \leq \\ &\leq (-m-1) \|L(\cdot)P_{[m+1, 0]}\|_\infty \|g\|_1 < \infty. \end{aligned} \quad (4.6)$$

From (4.3) and (4.4) we get

$$\sum_{n=0}^{\infty} |L(n)P_{[-\infty, m]}(Hg)(n)| \leq \|L(\cdot)P_{[-\infty, m]}\|_1 \|Hg\|_{\infty} \leq \|L(\cdot)P_{[-\infty, m]}\|_1 \|g\|_1 < \infty. \quad (4.7)$$

Combining (4.6) and (4.7), we see that $M_L Hg \in \ell^1(\mathcal{X})$ for every $g \in \ell^1(\mathcal{B}^0)$. From (ℓ^1, ℓ^∞) -stability we get $\Gamma M_L Hg \in \ell^\infty(\mathcal{X})$. \square

Remark 4.10. *Example 6.5 shows that condition (4.3) in Corollary 4.9 cannot be relaxed to condition $\|L(\cdot)P_{[-\infty, -1]}\|_p < \infty$ with $p > 1$.*

5 Subdiagonal systems and independence of stability properties of the parameter γ

The continuous embedding

$$\mathcal{B}^\gamma \subseteq \mathcal{B}^\delta, \quad |\varphi|_{\mathcal{B}^\delta} \leq |\varphi|_{\mathcal{B}^\gamma} \quad \text{for } -\infty < \gamma \leq \delta < \infty,$$

and the definitions of uniform stability and UE stability imply easily the following statement.

Proposition 5.1. *Let $\delta \in \mathbb{R}$. If system (2.3) is UES (US) in \mathcal{X} w.r.t. \mathcal{B}^δ , then for any $\gamma \in (-\infty, \delta)$ system (2.3) is UES (resp., US) in \mathcal{X} w.r.t. \mathcal{B}^γ .*

It is easy to see that the inverse implication is not true (for instance, from Example 6.6).

In this section we will show that under assumption (3.2) with $\gamma > 0$, uniform stability and UE stability in \mathcal{B}^δ do not depend on the choice of the parameter $\delta \in (0, \gamma]$, and, moreover, do not depend on the 'upper-triangular' part of the infinite operator-matrix $(L(n, j))_{n, j \geq 0}$.

Definition 5.2. *Assume that $L(n)$ is defined on \mathcal{B}_{fin} for all $n \in \mathbb{Z}^+$ and that $L(n, k) \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ for all $n, k \in \mathbb{Z}^+$. Then we will say that (1.1) is a subdiagonal system if*

$$L(n)P_{[-\infty, -n]}\varphi = 0_{\mathcal{X}}, \quad \varphi \in \text{Dom } L(n), \quad n \in \mathbb{Z}^+. \quad (5.1)$$

If (1.1) is a subdiagonal system, then it can be written in the form

$$x(n+1) = \sum_{j=1}^n L(n, n-j)x(j) + f(n), \quad n \in \mathbb{Z}^+, \quad (x(0) = 0_{\mathcal{X}} \text{ due to (2.1)}) \quad (5.2)$$

and it can be considered on the whole vector space $\mathcal{X}^{\mathbb{Z}^-}$.

Remark 5.3. *One can see that subdiagonal system is a particular case of Volterra difference system with unbounded delay (see (1.5)). Subdiagonal systems are characterized by the condition $L(n, n) = 0$.*

Having an arbitrary operators $L(n) : \text{Dom } L(n) \rightarrow \mathcal{X}$, $n \in \mathbb{Z}^+$, with domains satisfying $\mathcal{B}_{\text{fin}} \subseteq \text{Dom } L(n) \subseteq \mathcal{X}^{\mathbb{Z}^-}$, we define the operator-valued function L_{subd} on \mathbb{Z}^+ by

$$L_{\text{subd}}(0)\varphi = 0_{\mathcal{X}}, \quad \varphi \in \mathcal{X}^{\mathbb{Z}^-}, \quad (5.3)$$

$$L_{\text{subd}}(n)\varphi = L(n)P_{[-n+1, 0]}\varphi, \quad \varphi \in \mathcal{X}^{\mathbb{Z}^-}, \quad n \in \mathbb{N}. \quad (5.4)$$

If $L(n, k) \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ for all $n, k \in \mathbb{Z}^+$, then L_{subd} defines a subdiagonal system. For a function $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$ the assumption $L(n, k) \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ is always fulfilled and therefore the following definition is natural.

Definition 5.4. Let $\gamma \in \mathbb{R}$ and let a function $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$ define system (1.1). Then we will say that the system defined by the function L_{subd} is the subdiagonal system associated with system (1.1).

Definition 5.4 is justified by the following proposition.

Proposition 5.5. In the settings of Definition 5.4 system (1.1) is (ℓ^p, ℓ^q) -stable if and only if the associated subdiagonal system is (ℓ^p, ℓ^q) -stable.

For the proof it is enough to notice that Definition 2.3 implies that (ℓ^p, ℓ^q) -stability of system (1.1) does not depend on the parts $L(n)P_{[-\infty, -n]}$ of the operators $L(n)$.

Theorem 5.6. Let $\gamma > 0$ and let a function $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$ define system (1.1). Assume that (3.2) holds. Then the following statements are equivalent:

- (i) System (1.1) is UES (US) in \mathcal{B}^γ .
- (ii) The subdiagonal system associated with (1.1) is UES (resp., US) in \mathcal{B}^γ .
- (iii) For any $\delta > 0$ and any function $\tilde{L} : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^\delta, \mathcal{X})$ such that

$$\begin{aligned} \tilde{L}_{\text{subd}}(n) &= L_{\text{subd}}(n), \quad n \in \mathbb{Z}^+, \quad \text{and} \\ \sup_{n \in \mathbb{Z}^+} \|\tilde{L}(n)P_{[-\infty, n]}\|_{\mathcal{B}^\delta \rightarrow \mathcal{X}} &< \infty \quad \text{for a certain } l \in \mathbb{Z}^-, \end{aligned} \quad (5.5)$$

the system defined by $\tilde{L}(\cdot)$ is UES (resp., US) in \mathcal{B}^δ .

The theorem follows immediately from Proposition 5.5 and Theorem 3.1 (resp., Theorem 4.5).

Remark 5.7. 1) Taking the function L defined in Example 6.4, one can see that the system defined by L_{subd} is US in any of \mathcal{B}^γ with $\gamma > 0$, while the system defined by L is not US in \mathcal{B}^γ , $\gamma > 0$. So Example 6.4 shows that neither condition (3.2) nor (5.5) can be dropped in the US version of Theorem 5.6. For the UES case of Theorem 5.6 the same conclusion can be inferred from Example 6.3.

2) The systems introduced in Examples 6.4 and 6.3 satisfy condition (3.2) with $\gamma \leq 0$. This implies easily that in general Theorem 5.6 is not valid for $\gamma \leq 0$ in both the US and UES (in \mathcal{X}) versions.

6 Examples

In this section we assume that the Banach space \mathcal{X} is nontrivial, i.e., $\mathcal{X} \neq \{0_{\mathcal{X}}\}$. In particular, all the arguments below are valid if $\mathcal{X} = \mathbb{R}$ or $\mathcal{X} = \mathbb{C}$.

The following example shows that condition (3.2) cannot be omitted in Theorem 3.1 and Corollary 3.4.

Example 6.1. Let $1 \leq q < r \leq \infty$, $a(\cdot) \in l^r(\mathbb{R}) \setminus l^q(\mathbb{R})$. Let us define $L(n) : \mathcal{X}^{\mathbb{Z}^-} \rightarrow \mathcal{X}$ by $L(0) = 0$ and $L(n)\varphi = a(n)\varphi^{[-n+1]}$ for $n \in \mathbb{N}$. The function L defines the system

$$x(1) = f(0), \quad x(n+1) = a(n)x(1) + f(n), \quad n \in \mathbb{N}, \quad (6.1)$$

on all the spaces \mathcal{B}^γ with $\gamma \in \mathbb{R}$. It is easy to see that for any $p \leq r$ system (6.1) is (ℓ^p, ℓ^r) -stable, but it is *not* (ℓ^p, ℓ^q) -stable.

Indeed, considering the solution $x(n) = x(n, 0, 0_{\mathcal{B}}; f)$ with $f \in \ell^p$, we get $x(n+1) = a(n)f(0) + f(n)$, $n \in \mathbb{N}$. Since $a(\cdot) \in \ell^r$ and $f(\cdot) \in \ell^p \subseteq \ell^r$, we see that $x(\cdot) \in \ell^r$. On the other hand, it follows from $a(\cdot) \notin \ell^q$ that $x(\cdot) \notin \ell^q$ whenever $f(0) \neq 0$ and $f \in \ell^p \cap \ell^q$.

Note that

$$x(n+1, 1, \varphi; \mathbf{0}) = a(n)x(1, 1, \varphi; \mathbf{0}) = a(n)\varphi^{[0]}, \quad n \in \mathbb{N}.$$

Hence, $a(\cdot) \notin \ell^q$ implies that the homogeneous system associated with (6.1) is *not UES* in \mathcal{X} w.r.t. any of the spaces \mathcal{B}^γ , $\gamma \in \mathbb{R}$.

The next example shows that condition (3.2) cannot be replaced by the less restrictive condition (3.3) in Theorems 3.1 and 4.5.

Example 6.2. Let $1 \leq p \leq q \leq \infty$, $\gamma > 0$, and let \mathcal{B}^γ be the phase space. Let a sequence $m_n \in \mathbb{Z}^-$, $n \in \mathbb{Z}^+$, be such that $\liminf m_n = -\infty$. Then there exists $L : \mathbb{Z}^+ \rightarrow \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$ such that:

- (i) condition (3.3) is fulfilled,
- (ii) system (1.1) is (ℓ^p, ℓ^q) -stable,
- (iii) but $\sup_{n \in \mathbb{Z}^+} \|L(n)\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} = \infty$ and so system (2.3) is *neither UES nor US* in \mathcal{X} (due to Remark 3.2 and Proposition 4.4).

Let us construct the corresponding operator-function. Since $\liminf m_n = -\infty$, we can choose an increasing sequence $n_k \in \mathbb{Z}^+$, $k \in \mathbb{N}$, such that

$$n_k > n_{k-1} + k + 1, \quad (6.2)$$

$$m_{n_k} < -k. \quad (6.3)$$

Let us define $L(\cdot)$ by

$$L(n) = 0 \quad \text{if } n \notin \{n_k\}_1^\infty, \quad (6.4)$$

$$L(n_k)\varphi = \varphi^{[-k]}, \quad \varphi \in \mathcal{B}^\gamma, \quad k \in \mathbb{N}. \quad (6.5)$$

Then condition (3.3) is fulfilled since (6.4), (6.3) and (6.5) imply $L(n)P_{[-\infty, m_n]} = 0$.

Further, $\sup_{n \in \mathbb{Z}^+} \|L(n)\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} = \infty$. Indeed, (6.5) yields

$$\|L(n_k)\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} = \sup_{\varphi \neq 0_{\mathcal{B}}} \frac{|\varphi^{[-k]}|}{|\varphi|_{\mathcal{B}^\gamma}} = e^{k\gamma}, \quad \text{and therefore} \quad \lim_{k \rightarrow \infty} \|L(n_k)\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} = \infty.$$

On the other hand, system (1.1) is (ℓ^p, ℓ^q) -stable. Indeed, due to $p \leq q$ and (2.2), it is enough to show that $x(\cdot, 0, 0_{\mathcal{X}}; f) \in \ell^p$ for any $f(\cdot) \in \ell^p$. It is easy to see that

$$\begin{aligned} x(n+1, 0, 0_{\mathcal{X}}; f) &= f(n) \quad \text{if } n \notin \{n_k\}_1^\infty, \\ x(n+1, 0, 0_{\mathcal{X}}; f) &= x_{n_k}^{[-k]} + f(n_k) \quad \text{if } n = n_k. \end{aligned} \quad (6.6)$$

By definition, $x_{n_k}^{[-k]} = x(n_k - k)$. Taking into account (6.2), we see that $n_k - k > n_{k-1} + 1$ and so (6.6) implies $x_{n_k}^{[-k]} = f(n_k - k - 1)$. Hence,

$$x(n+1, 0, 0_{\mathcal{X}}; f) = f(n_k - k - 1) + f(n_k) \quad \text{if } n = n_k. \quad (6.7)$$

By (6.2) both sequences $\{n_k\}$ and $\{n_k - k - 1\}$ are strictly increasing. Thus, by Minkowski's inequality,

$$\|x(\cdot, 0, 0_{\mathcal{X}}; f)\|_p \leq 2\|f\|_p.$$

[4, Example 2] can be used to prove that neither condition (3.2) nor condition (5.5) can be dropped in the UES version of Theorem 5.6 and that for $\gamma \leq 0$ the implication (iii) \Rightarrow (i) of Theorem 3.1 is not valid in general.

Example 6.3 ([4]). Define $L(n)$ by $L(n)\varphi = \frac{1}{2}\varphi^{[0]} + \varphi^{[-n]}$, $n \in \mathbb{Z}^+$. The function L defines the system

$$x(n+1) = \frac{1}{2}x(n) + x(0) + f(n), \quad n \in \mathbb{Z}^+. \quad (6.8)$$

Let us show that for any $\gamma \leq 0$:

- (i) function L satisfies condition (3.2),
- (ii) system (6.8) is (ℓ^p, ℓ^q) -stable for any $1 \leq p \leq q \leq \infty$,
- (iii) but the homogeneous system associated with (6.8) *is not UES in \mathcal{X} w.r.t. \mathcal{B}^γ* .

Indeed, (3.2) is satisfied in its strongest form (i.e., with $m = 0$) since for $\gamma \leq 0$

$$|L(n)\varphi| \leq \frac{1}{2}|\varphi^{[0]}| + |\varphi^{[-n]}| \leq \left(\frac{1}{2} + e^{n\gamma}\right) |\varphi|_{\mathcal{B}^\gamma} \leq \frac{3}{2} |\varphi|_{\mathcal{B}^\gamma}.$$

By induction,

$$x(n, 0, 0_{\mathcal{B}}; f) = \sum_{k=0}^{n-1} 2^{-(n-k)+1} f(k), \quad n \in \mathbb{Z}^+.$$

Applying Young's inequality for convolutions (see e.g. [13, Problem VI.11.10]), one gets (ℓ^p, ℓ^q) -stability for $p \leq q$. It follows from $x(n, 0, \varphi; \mathbf{0}) = (2 - 2^{-n})x(0) = (2 - 2^{-n})\varphi^{[0]}$ that system $x(n+1) = \frac{1}{2}x(n) + x(0)$ is not UES in \mathcal{B}^γ .

The following example shows that in general Theorem 4.5 is not valid for $\gamma = 0$ and that neither condition (3.2) nor (5.5) can be dropped in the US version of Theorem 5.6.

Example 6.4 (cf. Example 1 in [4]). Operators $L(n)\varphi = \varphi^{[0]} + \varphi^{[-n]}$ define the system

$$x(n+1) = x(n) + x(0) + f(n), \quad n \in \mathbb{Z}^+. \quad (6.9)$$

It is easy to see that:

- (i) system (6.9) is (ℓ^1, ℓ^∞) -stable,
- (ii) condition (3.2) is fulfilled for $\gamma = 0$,
- (iii) but the homogeneous system associated with (6.9) *is not US in \mathcal{X} w.r.t. \mathcal{B}^0* (and so is not US in \mathcal{B}^0).

Indeed, induction shows that

$$x(n, 0, 0_{\mathcal{X}}; f) = \sum_{k=0}^{n-1} f(k), \quad n \in \mathbb{N}. \quad (6.10)$$

So (6.9) is (ℓ^1, ℓ^∞) -stable. For $\gamma = 0$ condition (3.2) is fulfilled with $m = 0$ since $\|L(n)\|_{\mathcal{B}^0 \rightarrow \mathcal{X}} = 2$. The system $x(n+1) = x(n) + x(0)$ is not US in \mathcal{X} since $x(n, 0, \varphi; \mathbf{0}) = (n+1)\varphi^{[0]}$.

From the next example, one can see that Condition (4.3) in Corollary 4.9 cannot be relaxed to condition $\|L(\cdot)P_{[-\infty, -1]}\|_p < \infty$ with $p > 1$.

Example 6.5 (cf. Example 3.1 in [11]). This is a development of Example 6.4. Let $1 < p \leq \infty$, $a(\cdot) \in l^p(\mathbb{N}) \setminus l^1(\mathbb{N})$ and $a(n) \geq 0$ for all $n \in \mathbb{N}$. Operators $L(n)\varphi = \varphi^{[0]} + a(n)\varphi^{[-n]}$ define the system

$$x(n+1) = x(n) + a(n)x(0) + f(n). \quad (6.11)$$

One can see that:

- (i) system (6.11) is (ℓ^1, ℓ^∞) -stable,
- (ii) $\|L(\cdot)P_{[-\infty, -1]}\|_p < \infty$,
- (iii) but the homogeneous system associated with (6.11) is not US in \mathcal{X} w.r.t. \mathcal{B}^0 .

Indeed, (6.11) is (ℓ^1, ℓ^∞) -stable since (6.10) still holds. From

$$|L(n)P_{[-\infty, -1]}\varphi| = |a(n)\varphi^{[-n]}| \leq a(n)|\varphi|_{\mathcal{B}^0},$$

one can get $\|L(\cdot)P_{[-\infty, -1]}\|_p < \infty$. The system $x(n+1) = x(n) + a(n)x(0)$ is not US in \mathcal{X} since

$$x(n, 0, \varphi; \mathbf{0}) = \left(1 + \sum_{k=0}^{n-1} a(k)\right) \varphi^{[0]}$$

and $\sum_{k=0}^{n-1} a(k) \rightarrow +\infty$ as $n \rightarrow \infty$.

The example below shows that the inverse implication in Proposition 5.1 is not true.

Example 6.6. Let $\delta > 0$. Consider the system

$$x(n+1) = ne^{-n\delta}x(0) + f(n). \quad (6.12)$$

Corresponding operators $L(n)$ are defined by $L(n)\varphi = ne^{-n\delta}\varphi^{[-n]}$. Then the homogeneous system associated with (6.12) is UES in \mathcal{X} w.r.t. \mathcal{B}^γ for all $\gamma \in (0, \delta)$ (and so for all $\gamma \in (-\infty, \delta)$ due to Proposition 5.1), but is not US in \mathcal{X} w.r.t. \mathcal{B}^δ .

Indeed, note that $x(\cdot+1, 0, 0_{\mathcal{X}}; f) = f(\cdot)$. Hence system (6.12) is (ℓ^p, ℓ^q) -stable for any $1 \leq p \leq q \leq \infty$. From this and Theorem 3.1 (Theorem 4.5), we see that system $x(n+1) = ne^{-n\delta}x(0)$ is UES (resp., US) in \mathcal{B}^γ with $\gamma > 0$ if and only if condition (3.2) is fulfilled. Using the specific form of $L(n)$, one obtains that for $n \geq -m$,

$$\|L(n)P_{[-\infty, m]}\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} = \|L(n)\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} = \sup_{\varphi \neq 0_{\mathcal{B}}} \frac{ne^{-n\delta}|\varphi^{[-n]}|}{|\varphi|_{\mathcal{B}^\gamma}} = ne^{-n\delta}e^{n\gamma}.$$

Thus, (3.2) is fulfilled exactly when $\gamma < \delta$.

7 Discussion and Open Problems

In the present paper we studied the connection of (ℓ^p, ℓ^q) -stability and uniform (exponential) stability. The paper comprehensively investigates the topic of Bohl-Perron type criteria for systems with infinite delay defined on the most popular family \mathcal{B}^γ , $\gamma > 0$, of fading phase spaces (for the definition of fading phase spaces see e.g. [21] and the discussion in the monograph [17]). However, there are still relevant open problems.

1. The choice of an appropriate seminormed or normed phase space \mathcal{B} for the system (1.1) is determined by the requirement $L(n) \in \mathcal{L}(\mathcal{B}, \mathcal{X})$ for all n . For the Volterra difference system (1.2) and phase spaces \mathcal{B}^γ this condition takes the form

$$\sum_{k=0}^{\infty} e^{k\gamma} \|L(n, k)\|_{\mathcal{X} \rightarrow \mathcal{X}} < \infty, \quad n \in \mathbb{Z}^+. \quad (7.1)$$

If we are interested in the question of uniform (exponential) stability in \mathcal{B}^γ ($\gamma > 0$), we also should take into account the fact that $\sup_{n \geq 0} \|L(n)\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} < \infty$ is a necessary condition (see Remark 3.2 and Proposition 4.4). In the case of Volterra system (1.2), condition $\sup_{n \geq 0} \|L(n)\|_{\mathcal{B}^\gamma \rightarrow \mathcal{X}} < \infty$ becomes

$$\sup_{n \geq 0} \sum_{k=0}^{\infty} e^{k\gamma} \|L(n, k)\|_{\mathcal{X} \rightarrow \mathcal{X}} < \infty. \quad (7.2)$$

On the other hand, having two phase spaces \mathcal{B}_1 and \mathcal{B}_2 with continuous embedding $\|\cdot\|_{\mathcal{B}_2} \leq \|\cdot\|_{\mathcal{B}_1}$, one can see that uniform (exponential) stability in \mathcal{X} w.r.t. \mathcal{B}_2 is a stronger property than uniform (exponential) stability in \mathcal{X} w.r.t. \mathcal{B}_1 (cf. Proposition 5.1).

If (7.1) or the stronger condition (7.2) is violated for all $\gamma > 0$, but (7.2) holds for $\gamma = 0$, then it is reasonable to consider the question of uniform (exponential) stability in \mathcal{X} w.r.t. \mathcal{B}^0 . However, the criteria of uniform and UE stabilities in fading phase spaces \mathcal{B}^γ with $\gamma > 0$ (Theorems 3.1 and 4.5) are *not valid* in the non-fading space \mathcal{B}^0 (see Examples 6.3 and 6.4).

We were not able to get a comprehensive description of uniform and UE stabilities in \mathcal{B}^0 in terms of (ℓ^p, ℓ^q) -stability. For uniform stability, we obtained a criterion of somewhat different type (Theorem 4.8), but one of its conditions has a complicated nature.

Problem 1. *Find a complete description of uniform and uniform exponential stabilities in the phase spaces \mathcal{B}^0 in terms of (ℓ^p, ℓ^q) -stability and certain boundedness conditions on the coefficients $L(n)$.*

Example 7.1. An example of a Volterra system that is UES in \mathcal{X} w.r.t. \mathcal{B}^0 and does not satisfy (7.1) for all $\gamma > 0$ is

$$x(n+1) = \sum_{k=-\infty}^n \frac{e^{-n}}{(n-k+1)(n-k+2)} x(k), \quad n \geq 0. \quad (7.3)$$

Indeed, it is easy to see by induction that $|x(\tau+k, \tau, \varphi; \mathbf{0})| \leq e^{-(\tau+k-1)} |\varphi|_{\mathcal{B}^0}$ for all $\tau \geq 0$ and $k \geq 1$.

Note that in [10] exponential stability is understood in the sense of exponentially decaying estimates on the fundamental matrix. This stability property does not depend on the choice of the phase space and, for the particular case of Volterra systems of the form (1.5), is stronger than UES stability in \mathcal{B}^γ for any $\gamma > 0$. The idea of [10] to consider (ℓ^p, ℓ^q) -stability with weighted ℓ^p -spaces may provide an approach to Problem 1.

In Corollary 4.9 we give two sufficient conditions of Bohl-Perron type for uniform stability in \mathcal{B}^0 . One can see that, in these conditions, the assumption on $L(n)$ is connected with the parameter p (which equals 1 or ∞) in (ℓ^p, ℓ^∞) -stability. The question is whether the interpolation to intermediate values of $p \in (1, \infty)$ also provides sufficient conditions.

Problem 2. *More precisely, for $p \in (1, \infty)$, prove or disprove that system (2.3) is US in \mathcal{B}^0 if (1.1) is (ℓ^p, ℓ^∞) -stable and*

$$\text{there exists } m \in \mathbb{Z}^- \text{ such that } \|L(\cdot)P_{[-\infty, m]}\|_p^p := \sum_{n=0}^{\infty} \|L(n)P_{[-\infty, m]}\|_{\mathcal{B}^0 \rightarrow \mathcal{X}}^p < \infty.$$

2. We used essentially the fact that the phase spaces \mathcal{B}^γ with $\gamma > 0$ are fading. Our results can be extended on wider class of phase spaces of ℓ^r type defined by

$$\mathcal{B}^{r,\gamma} := \{\varphi = \text{col}(\varphi^{[m]})_{m=-\infty}^0 : |\varphi|_{\mathcal{B}^{r,\gamma}} := \left(\sum_{m=-\infty}^0 |e^{\gamma m} \varphi^{[m]}|^r \right)^{1/r} < \infty\}, \quad 1 \leq r < \infty.$$

If $\gamma > 0$, the phase spaces $\mathcal{B}^{r,\gamma}$ are fading. We show below that our method works for this class.

Theorem 7.2. *Let $1 \leq r < \infty$ and $\gamma > 0$. Then Theorems 3.1 and 4.5 are valid with the phase space $\mathcal{B} = \mathcal{B}^{r,\gamma}$ instead of \mathcal{B}^γ .*

Proof. The general scheme of the proof of Theorems 3.1 and 4.5 works with minor changes in (2.15), (3.8), (3.20), (3.30), and more essential changes in Proposition 4.3 and in the proofs of Propositions 3.12 and 3.9. We explain here only essential changes.

Changes in the proof of Proposition 3.9. Estimate (3.18) can be adjusted to the phase space $\mathcal{B}^{r,\gamma}$ with the use of Minkowski's inequality:

$$\begin{aligned} |h(n)|_{\mathcal{B}^{r,\gamma}} &= \left(\sum_{m \in \mathbb{Z}^-} \left| \sum_{k=\max\{0, m+n\}}^{n-1} e^{(-n+k+1)\gamma} [e^{(m+n-k-1)\gamma} g^{[m+n-k-1]}(k)] \right|^r \right)^{1/r} \leq \\ &\leq \sum_{k=0}^{n-1} e^{(-n+k+1)\gamma} |g(k)|_{\mathcal{B}^{r,\gamma}} = (\mathfrak{e} * \mathfrak{g}_r)(n), \end{aligned}$$

where $\mathfrak{e}(\cdot)$ is given by (3.19) and $\mathfrak{g}_r(\cdot) := |g(\cdot)|_{\mathcal{B}^{r,\gamma}}$. The rest of the proof is the same.

Changes in the proof of Proposition 3.12. We use $|\varphi|_{\mathcal{B}^{r,\gamma}} \leq |P_{\{0\}}\varphi|_{\mathcal{B}^{r,\gamma}} + |(I - P_{\{0\}})\varphi|_{\mathcal{B}^{r,\gamma}}$ instead of (2.15). Then induction easily produce the inequality

$$|y(n, \tau, \varphi; \mathbf{0})|_{\mathcal{B}^{r,\gamma}} \leq (n - \tau + 1)K e^{-\nu_1(n-\tau)} |\varphi|_{\mathcal{B}^{r,\gamma}}.$$

Replacing ν_1 with any $\nu_2 \in (0, \nu_1)$ and changing $(n - \tau + 1)K$ to large enough constant K_1 , one gets UE stability in $\mathcal{B}^{r,\gamma}$.

For phase spaces $\mathcal{B}^{r,\gamma}$ Proposition 4.3 is valid only when $\gamma > 0$, the proof for this case requires the following changes. We again use $|\varphi|_{\mathcal{B}^{r,\gamma}} \leq |P_{\{0\}}\varphi|_{\mathcal{B}^{r,\gamma}} + |(I - P_{\{0\}})\varphi|_{\mathcal{B}^{r,\gamma}}$ instead of (2.15) and get by induction the inequality

$$|y(n, \tau, \varphi; \mathbf{0})|_{\mathcal{B}^{r,\gamma}} \leq K \left(\sum_{j=0}^{n-\tau} e^{-j\gamma} \right) |\varphi|_{\mathcal{B}^{r,\gamma}}.$$

This implies uniform stability in $\mathcal{B}^{r,\gamma}$ with $K_1 = K(1 - e^{-\gamma})^{-1}$. □

The non-fading memory spaces $\mathcal{B}^{r,0}$ appear naturally in the theory of Volterra difference systems (1.2) when the coefficients $L(n, k)$ satisfy condition of the $\ell^{r'}$ type:

$$\sum_{k=0}^{+\infty} \|L(n, k)\|_{\mathcal{X} \rightarrow \mathcal{X}}^{r'} < \infty \quad \left(\text{or, for } r = 1, \sup_{k \geq 0} \|L(n, k)\|_{\mathcal{X} \rightarrow \mathcal{X}} \right), \quad n \in \mathbb{Z}^+$$

(it is assumed that $1/r + 1/r' = 1$). Note that $\mathcal{B}^{r,0}$ compose a scale between $\bigcup_{\gamma < 0} \mathcal{B}^\gamma$ and \mathcal{B}^0 in the sense that the embedding

$$\mathcal{B}^\gamma \subseteq \mathcal{B}^{r,0} \subseteq \mathcal{B}^0, \quad |\varphi|_{\mathcal{B}^0} \leq |\varphi|_{\mathcal{B}^{r,0}} \leq (1 - e^{\gamma r})^{-1/r} |\varphi|_{\mathcal{B}^\gamma}, \quad \gamma < 0, \quad 1 \leq r < \infty, \quad (7.4)$$

holds as well as the usual ℓ^r embedding of spaces $\mathcal{B}^{r,0}$.

Problem 3. *Find Bohl-Perron type criterions for phase spaces $\mathcal{B}^{r,0}$. This question widens frames of Problem 1.*

3. Bohl-Perron type theorems were used in [4, Sec. 5] to derive explicit (i.e., given in the terms of coefficients) exponential stability tests for systems with bounded delay. It is interesting to apply the method of [4, Sec. 5] to Volterra difference systems with unbounded and infinite delay. This is a part of the following general problem.

Problem 4. *Find explicit tests of exponential stability (complementing the known ones) for Volterra difference systems with unbounded or infinite delay.*

One of the motivations for this question is that exponential stability (and, more generally, dichotomy) is used in the study of bounded solutions to nonlinear perturbations of Volterra difference equations (see e.g. [7, 23]).

On the other hand, there is not much literature devoted to explicit conditions of exponential stability in the cases of unbounded and infinite delay. For systems of convolution type

$$x(n+1) = \sum_{k=0}^n B(n-k)x(k) \quad (7.5)$$

some sufficient conditions can be derived from known results on asymptotical stability (see e.g. [14, 18]) with the use of [16, Theorem 5]. ([16, Theorem 5] gives a necessary and sufficient condition of exponential stability of (7.5) under the assumption that (7.5) is asymptotically stable.)

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